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Witt and cohomological invariants of Witt classes

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We describe all Witt invariants and mod 2 cohomological invariants of the functor I^n as combinations of fundamental invariants; this is related to the study of operations on mod 2 Milnor K-theory. We also study behaviour of these invariants with respect to products, restrictions, similitudes and ramification.

Introduction

Building on classical constructions such as the discriminant and the Hasse–Witt invariant, cohomological invariants have become a standard tool in the study of quadratic forms. Cohomological invariants of quadratic forms are also related to cohomological invariants of algebraic groups, for split groups of orthogonal type.

In [Garibaldi et al. 2003], Serre introduces cohomological invariants over a field, and completely describes (away from characteristic 2) the invariants of Quad_n (nondegenerated n-dimensional quadratic forms) and $\mathrm{Quad}_{n,\delta}$ (those with prescribed determinant δ), and in particular this settles the case of invariants of split orthogonal and special orthogonal groups. In contrast, the case of split spin groups, corresponding to invariants of $\mathrm{Quad}_n \cap I^3$ (meaning that the Witt classes of the forms must be in I^3), is very much open, and has only been treated for small n (see for instance [Garibaldi 2009]) or for invariants of small degree (the case of degree 3 has been essentially solved by Merkurjev [2016]), one problem being that we do not have any satisfying parametrization of $\mathrm{Quad}_n \cap I^3$.

On the other hand, if we move from isometry classes to Witt classes, following the resolution of Milnor's conjecture by Voevodsky, we have at hand good descriptions of I^n (see for instance [Elman et al. 2008]), and at least one important cohomological invariant of I^n, $e_n : I^n(K) \to H^n(K, \mu_2)$. The goal of this article is to describe all mod 2 cohomological invariants of I^n, and study some of their basic properties.

Our starting point is a construction of Rost [1999], who defines a certain natural operation $P_n : I^n(K) \to I^{2n}(K)$ which behaves like a divided square in the sense that $P_n\big(\sum \varphi_i\big) = \sum_{i<j} \varphi_i \cdot \varphi_j$ if φ_i are n-fold Pfister forms. After composing with

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e_{2n} this gives a cohomological invariant of I^n of degree $2n$. We generalize this to operations $\bar{\pi}_n^d : I^n \to I^{dn}$ for all $d \in \mathbb{N}$ and thus cohomological invariants of degree dn. Since our constructions involve both Witt invariants and cohomological invariants, in order to avoid repeating very similar proofs in both settings, we choose to adopt a unified point of view and treat both cases simultaneously, using A to denote either the Witt ring or mod 2 cohomology.

We define two sets of generators for invariants, f_n^d (the invariants mentioned above, see Proposition 2.2) and g_n^d (Definition 4.4), each being useful depending on the situation. The invariants g_n^d have the important property that only a finite number of them are nonzero on a fixed form (Proposition 4.7), which allows us to take infinite combinations, and we show that any invariant of I^n is equal to such a combination (Theorem 4.9). They are also better behaved with respect to similitudes (Proposition 7.6). On the other hand, the f_n^d are preferable for handling products (Proposition 5.2 and Corollary 5.6) and restriction to I^{n+1} (Corollaries 6.3 and 6.4). We also study behaviour with respect to residues from discrete valuations (Proposition 8.1), and establish links with Serre's description of invariants of isometry classes (Proposition 9.5).

Our invariants may be related to other various constructions on Milnor K-theory and Galois cohomology, notably by Vial [2009]. The invariants defined here may be seen as lifting of Vial's to the level of I^n. See Section 10 for more details.

Finally, we adapt an idea of Rost [1999] (see also [Garibaldi 2009]) to study invariants of Witt classes in I^n that are divisible by an r-fold Pfister form, giving a complete description for $r = 1$ (Theorem 11.4).

Notations and some preliminaries

In all that follows, k is a fixed field of characteristic different from 2, and K denotes any field extension of k. The set of natural integers is denoted by $\mathbb{N}$, and the positive integers by $\mathbb{N}^*$; if $x \in \mathbb{R}$, $\lfloor x \rfloor \in \mathbb{Z}$ denotes its floor, and $\lceil x \rceil$ its ceiling. We extend the binomial coefficient $\binom{a}{b}$ for arbitrary $a, b \in \mathbb{Z}$ in the only way that still satisfies Pascal's triangle.

For all facts on quadratic forms, the reader is referred to [Elman et al. 2008]. All the quadratic forms we consider are assumed to be nondegenerated. The Grothendieck–Witt ring $\mathrm{GW}(K)$ has a fundamental ideal $\hat{I}(K)$, defined as the kernel of the dimension map $\mathrm{GW}(K) \to \mathbb{Z}$. We denote by $[q] \in W(K)$ the Witt class of an element $q \in \mathrm{GW}(K)$, and this ring morphism $\mathrm{GW}(K) \to W(K)$ induces an isomorphism between $\hat{I}(K)$ and the fundamental ideal $I(K) \subset W(K)$. If $x \in I(K)$, we write $\hat{x} \in \hat{I}(K)$ for its (unique) antecedent. If $n \in \mathbb{N}$ and $q \in W(K)$, then $nq = q + \cdots + q$ is not to be confused with $\langle n \rangle q$, which is pointwise multiplication by the scalar $n \in K^*$.

If $a \in K^*$, then we write $\langle\langle a\rangle\rangle = \langle 1, -a\rangle \in I(K)$, and if $a_1, \dots, a_n \in K^*$, then $\langle\langle a_1, \dots, a_n\rangle\rangle = \langle\langle a_1\rangle\rangle \cdots \langle\langle a_n\rangle\rangle \in I^n(K)$. Those elements are (the Witt classes of) the n-fold Pfister forms, and we use $\mathrm{Pf}_n(K) \subset I^n(K)$ for the set of such elements. We also write $\langle|a_1, \dots, a_n|\rangle$ for the antecedent of $\langle\langle a_1, \dots, a_n\rangle\rangle$ in $\hat{I}^n(K)$; we call such elements n-fold Grothendieck–Pfister elements, and we write $\widehat{\mathrm{Pf}}_n(K) \subset \hat{I}^n(K)$ for their set. For instance, $\langle|a|\rangle = \langle 1\rangle - \langle a\rangle$, so $\langle|1|\rangle = 0$. Notice that if $q \in W(K)$, then $2q = \langle\langle -1\rangle\rangle q$, and in particular if -1 is a square in K then $2q = 0$ in $W(K)$. Also, if $\varphi \in \mathrm{Pf}_n(K)$, then $\varphi^2 = 2^n\varphi$, since $\langle\langle a, a\rangle\rangle = \langle\langle -1, a\rangle\rangle = 2\langle\langle a\rangle\rangle$. This relation is also true if $\varphi \in \widehat{\mathrm{Pf}}_n(K)$.

By a filtered group A we mean that there are subgroups $A^{\geqslant d}$ for all $d \in \mathbb{Z}$, such that $A^{\geqslant d+1} \subset A^{\geqslant d}$. We say the filtration is *positive* if $A^{\geqslant d} = A$ for all $d \leqslant 0$, and that it is *separated* if $\bigcap_d A^{\geqslant d} = 0$. If A is a ring, it is a filtered ring if

$$A^{\geqslant d} \cdot A^{\geqslant d'} \subset A^{\geqslant d+d'},$$

and M is a filtered A-module if it is a filtered group such that $A^{\geqslant d} \cdot M^{\geqslant d'} \subset M^{\geqslant d+d'}$. For any $n \in \mathbb{Z}$, we denote by $M[n]$ the filtered module such that $(M[n])^{\geqslant d} = M^{\geqslant d+n}$. A morphism of filtered modules $f : M \to N$ is a module morphism such that $f(M^{\geqslant d}) \subset N^{\geqslant d}$.

Let $\mathrm{Fields}_{/k}$ be the category of field extensions of k. If we are given functors $T : \mathrm{Fields}_{/k} \to \mathrm{Sets}$ and $A : \mathrm{Fields}_{/k} \to \mathrm{Ab}$ (the category of abelian groups), then an invariant of T with values in A (over k) is a natural transformation from T to A. The set of such invariants is naturally an abelian group, denoted $\mathrm{Inv}(T, A)$. If T takes values in *pointed* sets, then we can define *normalized* invariants as the ones that send the distinguished element to 0. This subgroup is denoted $\mathrm{Inv}_0(T, A)$, and we have $\mathrm{Inv}(T, A) = A(k) \oplus \mathrm{Inv}_0(T, A)$.

Since we want to unify proofs for Witt and cohomological invariants, we use $A(K)$ for either $W(K)$ or $H^*(K, \mu_2)$, writing $A = W$ or $A = H$ if we want to distinguish cases. For $d \in \mathbb{N}$, we set $A^{\geqslant d}(K) = I^d(K)$ if $A = W$, and $A^{\geqslant d}(K) = \bigoplus_{i \geqslant d} H^i(K, \mu_2)$ if $A = H$. Then $A(K)$ is a filtered $A(k)$-algebra, and the filtration is separated and positive. Note that according to the resolution of Minor's conjecture by Voevodsky et al., the graded ring associated to $A(K)$ is in both cases the mod 2 cohomology ring $H^*(K, \mu_2)$.

For any $n \in \mathbb{N}^*$, we write $M(n) = \mathrm{Inv}(I^n, A)$, and $M^{\geqslant d}(n) = \mathrm{Inv}(I^n, A^{\geqslant d})$ for all $d \in \mathbb{N}$. Similarly, the subgroups of normalized invariants are denoted $M_0(n)$ and $M_0^{\geqslant d}(n)$. Then $M(n)$ is a filtered $A(k)$-algebra, and $M_0(n)$ is a submodule.

We list here the formal properties of A on which the article relies. We have a group morphism $f_n : I^n(K) \to A^{\geqslant n}(K)$ (either the identity if $A = W$, or the morphism e_n given by the Milnor conjecture if $A = H$) and we write

$$\{a_1, \dots, a_n\} = f_n(\langle\langle a_1, \dots, a_n\rangle\rangle)$$

(so it is either a Pfister form or a Galois symbol, depending on A). Note that

$$f_n(x)\cdot f_m(y)=f_{n+m}(xy). \tag{0.1}$$

We set $\delta=\delta(A)=1$ if $A=W$, and $\delta=0$ if $A=H$. Then we have

$$\forall a,b\in K^*,\ \{ab\}=\{a\}+\{b\}-\delta\{a,b\} \tag{0.2}$$

and

$$\delta\{-1\}=2\in A(K). \tag{0.3}$$

We also freely use the following lemmas:

Lemma 0.4. *If $x\in A(K)$ is such that for any extension L/K and any $\varphi\in \mathrm{Pf}_n(L)$ we have $f_n(\varphi)\cdot x\in A^{\geqslant d+n}(L)$, then $x\in A^{\geqslant d}(K)$. In particular, for any $n\in\mathbb{N}^*$, if $f_n(\varphi)\cdot x=0$ for all $\varphi\in\mathrm{Pf}_n(L)$, then $x=0$.*

Lemma 0.5. $\mathrm{Inv}(\mathrm{Pf}_n, A)=A(K)\oplus A(K)\cdot f_n$, *where we consider invariants defined over K.*

The first lemma can be proved by specialization, taking φ to be a generic Pfister form; the second corresponds to two theorems of Serre [Garibaldi et al. 2003, Theorem 18.1, Example 27.17].

1. Some pre-λ-ring structures

We refer to [Yau 2010] for the basic theory of λ-rings. If R is a commutative ring, a pre-λ-ring structure on R is the data of applications $\lambda^d: R\to R$ for all $d\in\mathbb{N}$ such that for all $x,y\in R$,

(i) $\lambda^0(x)=1$;

(ii) $\lambda^1(x)=x$;

(iii) $\forall d\in\mathbb{N},\ \lambda^d(x+y)=\sum_{k=0}^d\lambda^k(x)\lambda^{d-k}(y)$.

Example 1.1. The example we are interested in is $R=\mathrm{GW}(K)$. The λ^d are the exterior powers of bilinear forms, as defined in [Bourbaki 1970], and it is shown in [McGarraghy 2002] that they define a λ-ring structure on $\mathrm{GW}(K)$ (which is a pre-λ-ring structure with additional conditions).

We define $\Lambda(R)=1+tR[\![t]\!]$, the subset of formal power series with coefficients in R that have a constant coefficient equal to 1. It is a group for the multiplication of formal series. If we set $\lambda_t(x)=\sum_{d\in\mathbb{N}}\lambda^d(x)t^d\in R[\![t]\!]$, we see that a pre-λ-ring structure on R is equivalent to the data of a group morphism $\lambda_t:(R,+)\to(\Lambda(R),\cdot)$ such that for all $x\in R$ the degree 1 coefficient of $\lambda_t(x)$ is x. We switch freely between those two descriptions.

Example 1.2. For the canonical λ-ring structure on $\mathrm{GW}(K)$, we have $\lambda_t(\langle a\rangle)=1+\langle a\rangle t$ for all $a\in K^*$.

Recall that for any formal series $f, g \in R[[t]]$ such that the constant coefficient of f is zero, we can define the composition $g \circ f \in R[[t]]$. If furthermore the degree 1 coefficient of f is invertible in R, then f has an inverse for the composition, which we denote $f^{\circ -1}$.

Lemma 1.3. *Let R be a commutative ring. If $\lambda_t : R \to \Lambda(R)$ defines a pre-λ-ring structure on R, then for any $f \in t + t^2 R[[t]]$, the map*

$$\lambda_{f(t)} : R \to \Lambda(R), \quad x \mapsto \lambda_t(x) \circ f = \sum_{d \in \mathbb{N}} \lambda^d(x) f(t)^d$$

also defines a pre-λ-ring structure.

Proof. We have for any $x, y \in R$

$$\lambda_t(x+y) \circ f = (\lambda_t(x)\lambda_t(y)) \circ f = (\lambda_t(x) \circ f) \cdot (\lambda_t(y) \circ f).$$

Furthermore, since the degree 1 term of $f(t)$ is t, the degree 1 coefficient of $\lambda_t(x) \circ f$ is the same as that of $\lambda_t(x)$, which is x. □

We want to define for each $n \in \mathbb{N}^*$ a pre-λ-ring structure on $\mathrm{GW}(K)$ that vanishes for $d \geqslant 2$ on n-fold Grothendieck–Pfister elements. Our starting point is the following fundamental observation.

Lemma 1.4. *Let $a \in K^*$. For any $d \geqslant 1$, we have $\lambda^d(\langle\langle a \rangle\rangle) = \langle\langle a \rangle\rangle$. Therefore,*

$$\lambda_t(\langle\langle a \rangle\rangle) = 1 + \langle\langle a \rangle\rangle \theta(t),$$

where $\theta(t) = \sum_{d \geqslant 1} t^d = t/(1-t)$.

Proof. We have

$$\lambda_t(1 - \langle a \rangle) = \frac{\lambda_t(1)}{\lambda_t(\langle a \rangle)} = \frac{1+t}{1 + \langle a \rangle t} = 1 + \sum_{d \geqslant 1} (1 - \langle a \rangle) t^d,$$

using $\langle a \rangle^2 = 1$. □

We then define some formal series: for any $n \in \mathbb{N}^*$, $x_n(t) \in \mathbb{Z}[[t]]$ is defined recursively by

$$x_1(t) = \theta(t) = \frac{t}{1-t}, \quad x_{n+1} = x_n + 2^{n-1} x_n^2,$$

and $h_n(t) \in \mathbb{Q}[[t]]$ by

$$h_n = x_n^{\circ -1}.$$

Lemma 1.5. *For any $n \in \mathbb{N}^*$, we have $h_n(t) \in \mathbb{Z}[[t]]$. Furthermore, if a_n and b_n are the even part and odd part of x_n, respectively, then*

$$\begin{cases} a_{n+1} = 2^n b_n^2 = 2a_n + 2^n a_n^2, \\ b_{n+1} = b_n + 2^n a_n b_n. \end{cases}$$

Proof. Note first that $h_1(t) = t/(1+t) \in \mathbb{Z}[[t]]$. Let $p_n(t) = t + 2^{n-1}t^2 \in \mathbb{Z}[[t]]$; then by definition $x_{n+1} = p_n \circ x_n$, so $h_{n+1} = h_n \circ p_n^{\circ -1}$. Now a simple computation yields $p_n^{\circ -1} = tC(-2^{n-1}t)$, where $C(t) = (1 - \sqrt{1-4t})/2t$ is the generating function of the Catalan numbers (this is essentially equivalent to the well-known functional equation for $C(t)$); in particular, $p_n^{\circ -1}$ has integer coefficients, so $h_n(t) \in \mathbb{Z}[[t]]$.

Separating even and odd parts, the recursive definition of x_n yields

$$\begin{cases} a_{n+1} = a_n + 2^{n-1}a_n^2 + 2^{n-1}b_n^2, \\ b_{n+1} = b_n + 2^n a_n b_n. \end{cases}$$

So we need to show that for any $n \in \mathbb{N}^*$, $a_n + 2^{n-1}a_n^2 = 2^{n-1}b_n^2$. If $n = 1$, this is a direct computation, using $a_1(t) = t^2/(1-t^2)$ and $b_1(t) = t/(1-t^2)$.

Now suppose the formula holds until $n \in \mathbb{N}^*$. Then

$$\begin{aligned} a_{n+1} + 2^n a_{n+1}^2 &= 2^n b_n^2 + 2^n(2^n b_n^2)^2 = 2^n b_n^2(1+2^{2n}b_n^2), \\ 2^n b_{n+1}^2 = 2^n b_n^2(1+2^n a_n)^2 &= 2^n b_n^2(1+2^{n+1}a_n + 2^{2n}a_n^2) = 2^n b_n^2(1+2^{2n}b_n^2), \end{aligned}$$

which shows the expected formula. □

We can now use those formal series to define our pre-λ-ring structures.

Theorem 1.6. *For any $n \in \mathbb{N}^*$, the map $(\pi_n)_t = \lambda_{h_n(t)}$ defines a pre-λ-ring structure on $\mathrm{GW}(K)$ such that $\pi_n^d(\varphi) = 0$ for any $\varphi \in \widehat{\mathrm{Pf}}_n(K)$ and any $d \geqslant 2$.*

Proof. According to Lemma 1.3, $(\pi_n)_t$ does define a pre-λ-ring structure on $\mathrm{GW}(K)$. We show the statement about Grothendieck–Pfister elements by induction on n. For $n = 1$, the statement is equivalent to Lemma 1.4 since for any $\varphi \in \widehat{\mathrm{Pf}}_1(K)$, $\lambda_t(\varphi) = 1 + \varphi x_1(t)$ and $h_1 = x_1^{\circ -1}$.

Suppose the statement holds until $n \in \mathbb{N}^*$. Let $\varphi \in \widehat{\mathrm{Pf}}_{n+1}(K)$, and write $\varphi = \langle\langle a \rangle\rangle \psi$ with $a \in K^*$ and $\psi \in \widehat{\mathrm{Pf}}_n(K)$. We then need to show $\lambda_{h_{n+1}(t)}(\varphi) = 1 + \varphi t$, which is equivalent to

$$\lambda_t(\langle\langle a \rangle\rangle \psi) = 1 + \langle\langle a \rangle\rangle \psi x_{n+1}(t).$$

Note that for any $x \in \hat{I}(K)$, we have $-\langle a \rangle x = \langle -a \rangle x$, which implies that $\lambda^d(-\langle a \rangle x) = (-1)^d \langle a^d \rangle \lambda^d(x)$ for any $d \in \mathbb{N}$, and thus $\lambda_t(-\langle a \rangle x) = \lambda_{-\langle a \rangle t}(x)$. Therefore, we have in $\mathrm{GW}[[t]]$

$$\begin{aligned} \lambda_t(\psi - \langle a \rangle \psi) &= \lambda_t(\psi)\lambda_{-\langle a \rangle t}(\psi) \\ &= (1 + \psi x_n(t))(1 + \psi x_n(-\langle a \rangle t)) \\ &= 1 + \psi\big(x_n(t) + x_n(-\langle a \rangle t) + 2^n x_n(t) x_n(-\langle a \rangle t)\big). \end{aligned}$$

Thus we can conclude if we show that

$$x_n(t) + x_n(-\langle a \rangle t) + 2^n x_n(t) x_n(-\langle a \rangle t) = (1 - \langle a \rangle) x_{n+1}(t).$$

If we decompose in even and odd parts, this amounts to

$$\begin{cases} a_n(t) + a_n(t) + 2^n(a_n(t)^2 - \langle a\rangle b_n(t)^2) = (1-\langle a\rangle)a_{n+1}(t), \\ b_n(t) - \langle a\rangle b_n(t) + 2^n(b_n(t)a_n(t) - \langle a\rangle a_n(t)b_n(t)) = (1-\langle a\rangle)b_{n+1}(t), \end{cases}$$

which are consequences of Lemma 1.5. □

Remark 1.7. Those are *not* λ-ring structures; for instance, $(\pi_1)_t(1) = 1 + h_1(t) = 1 + t - t^2 + \cdots$, so $\pi_1^d(1) \neq 0$ for all $d \geqslant 2$.

Corollary 1.8. *Let* $n \in \mathbb{N}^*$, *and* $\varphi_1, \ldots, \varphi_r \in \widehat{\mathrm{Pf}}_n(K)$. *Then*

$$\pi_n^d\left(\sum_{i=1}^r \varphi_i\right) = \sum_{1\leqslant i_1<\cdots<i_d\leqslant r} \varphi_{i_1}\cdots\varphi_{i_d}.$$

In particular, $\pi_n^d(\hat{I}^n(K)) \subset \hat{I}^{nd}(K)$, *and* π_n^d *is zero on forms that are sums of* $d-1$ (*or less*) *n-fold Grothendieck–Pfister elements.*

Proof. The formula is proved by an easy induction, exactly similar to the proof of the formula for exterior powers of diagonal quadratic forms (or more generally λ-powers of a sum of elements of dimension 1 in any pre-λ-ring). If $x \in \hat{I}^n(K)$, then $x = x_1 - x_2$, where the x_i are sums of elements of $\widehat{\mathrm{Pf}}_n(K)$, and $(\pi_n)_t(x) = (\pi_n)_t(x_1)\cdot((\pi_n)_t(x_2))^{-1}$. Now it is easy to see that since the degree d coefficient of $(\pi_n)_t(x_i)$ is in $\hat{I}^{nd}(K)$, then the same is true for $(\pi_n)_t(x)$. □

Note that the formula in Corollary 1.8 is not enough to completely describe π_n^d on $\hat{I}^n(K)$, even if we could show directly that it is well-defined (which is possible using the presentation of $I^n(K)$ given in [Elman et al. 2008, Theorem 42.4]), since not every element of $I^n(K)$ is a sum of Pfister forms.

The idea of similar "divided power" operations on related structures such as Milnor K-theory of Galois cohomology has been around for some time (see Section 10 for more details).

2. The fundamental invariants

We now use these various pre-λ-ring structures on $\mathrm{GW}(K)$ to define some invariants of I^n.

Definition 2.1. Let $n \in \mathbb{N}^*$ and $d \in \mathbb{N}$. Then we define

$$f_n^d : I^n(K) \xrightarrow{\sim} \hat{I}^n(K) \xrightarrow{\pi_n^d} \hat{I}^{nd}(K) \xrightarrow{\sim} I^{nd}(K) \xrightarrow{f_{nd}} A^{\geqslant nd}(K).$$

If $A = W$, then we sometimes write $f_n^d = \bar{\pi}_n^d$.
If $A = H$, then we sometimes write $f_n^d = u_{nd}^{(n)}$.

This is well-defined according to Corollary 1.8. The notation $u_{nd}^{(n)}$ may seem dissonant with the rest, but we chose to stick with the tradition to write the degree of cohomological invariants in the index, and the exponent serves to distinguish between, for instance, $u_6^{(2)} : I^2(K) \to H^6(K,\mu_2)$ and $u_6^{(3)} : I^3(K) \to H^6(K,\mu_2)$, which are completely different ($u_6^{(3)}$ is *not* the restriction of $u_6^{(2)}$ to I^3).

Proposition 2.2. *Let $n \in \mathbb{N}^*$. Then for any $d \in \mathbb{N}$, we have $f_n^d \in M^{\geqslant nd}(n)$, and $(f_n^d)_{d\in\mathbb{N}}$ is the only family of elements of $M(n)$ such that*

(i) *$f_n^0 = 1$ and $f_n^1 = f_n$;*

(ii) *for all $q, q' \in I^n(K)$,*

$$f_n^d(q+q') = \sum_{k=0}^{d} f_n^k(q) \cdot f_n^{d-k}(q');$$

(iii) *for all $\varphi \in \mathrm{Pf}_n(K)$ and $d \geqslant 2$, $f_n^d(\varphi) = 0$.*

Furthermore, for any $\varphi \in \mathrm{Pf}_n(K)$ and any $d \in \mathbb{N}^*$,

$$f_n^d(-\varphi) = (-1)^d \{-1\}^{n(d-1)} f_n(\varphi). \tag{2.3}$$

Proof. The fact that f_n^d is an invariant is clear by construction: the definition of π_n^d is made in terms of the exterior powers, which are of course compatible with field extensions, and the expression of the π_n^d in terms of the λ^d is given by a universal $h_n \in \mathbb{Z}[\![t]\!]$.

The three properties are direct consequences of Theorem 1.6, after applying f_{nd} to the corresponding formulas for π_n^d (and using formula (0.1)).

The last formula on opposites of Pfister forms can be easily proved by induction using

$$0 = f_n^d(\varphi - \varphi) = f_n^d(-\varphi) + f_n^{d-1}(-\varphi) f_n(\varphi).$$

Uniqueness follows from property (ii) and the fact that Pfister forms additively generate $I^n(K)$, since the values of f_n^d are fixed on $\pm\varphi$ for any $\varphi \in \mathrm{Pf}_n(K)$. □

The following corollary is an immediate consequence of either Corollary 1.8 or Proposition 2.2.

Corollary 2.4. *Let $n \in \mathbb{N}^*$ and $\varphi_1, \dots, \varphi_r \in \mathrm{Pf}_n(K)$. Then*

$$f_n^d\left(\sum_{i=1}^{r} \varphi_i\right) = \sum_{1\leqslant i_1<\cdots<i_d\leqslant r} f_n(\varphi_{i_1}) \cdots f_n(\varphi_{i_d}).$$

In particular, f_n^d is zero on forms that are sums of $d-1$ or less n-fold Pfister forms.

3. The shifting operator

Since $I^n(K)$ is additively generated by the n-fold Pfister forms, it is natural to study how the invariants behave under adding or subtracting a Pfister form.

Proposition 3.1. *Let $n \in \mathbb{N}^*$ and $\varepsilon = \pm 1$. There is a unique morphism of filtered $A(k)$-modules $\Phi_n^\varepsilon : M(n) \to M(n)[-n]$ such that*

$$\alpha(q + \varepsilon\varphi) = \alpha(q) + \varepsilon f_n(\varphi) \cdot \Phi_n^\varepsilon(\alpha)(q)$$

for all $\alpha \in M(n)$, $q \in I^n(K)$ and $\varphi \in \mathrm{Pf}_n(K)$.

Proof. Let $\alpha \in M(n)$ and $q \in I^n(K)$. For any extension L/K and any $\varphi \in \mathrm{Pf}_n(L)$, we set

$$\beta_q(\varphi) = \alpha(q + \varepsilon\varphi).$$

Then $\beta_q \in \mathrm{Inv}(\mathrm{Pf}_n, A)$, defined over K. According to Lemma 0.5, there are uniquely determined $x_q, y_q \in A(K)$ such that $\beta_q = x_q + y_q \cdot f_n$.

Taking $\varphi = 0$ we see that $x_q = \alpha(q)$, and we then set $\Phi_n^\varepsilon(\alpha)(q) = \varepsilon y_q$, which gives the expected formula, as well as the uniqueness of Φ_n^ε.

By definition, Φ_n^ε is clearly an $A(k)$-module morphism, and it is of degree $-n$ because if $\alpha \in M^{\geqslant d}(n)$, then for any $q \in I^n(K)$, $f_n(\varphi) \cdot \alpha^\varepsilon(q) \in A^{\geqslant d}(L)$ for all $\varphi \in \mathrm{Pf}_n(L)$ and any extension L/K. Thus $\alpha^\varepsilon(q) \in A^{\geqslant d-n}(K)$ by Lemma 0.4. □

We often write $\Phi^+ = \Phi_n^{+1}$ and $\Phi^- = \Phi_n^{-1}$, as there is in practice no confusion to what n is in the context. We also write $\alpha^+ = \Phi^+(\alpha)$ and $\alpha^- = \Phi^-(\alpha)$ for any $\alpha \in M(n)$. These two operators have natural links between each other:

Proposition 3.2. *Let $n \in \mathbb{N}^*$. The operators Φ_n^+ and Φ_n^- commute, and furthermore, for any $\alpha \in M(n)$ we have*

$$\alpha^+ - \alpha^- = \{-1\}^n \alpha^{+-} = \{-1\}^n \alpha^{-+}.$$

Proof. Let $q \in I^n(K)$ and $\varphi, \psi \in \mathrm{Pf}_n(L)$. We have

$$\begin{aligned}\alpha(q + \varphi - \psi) &= \alpha(q + \varphi) - f_n(\psi)\alpha^-(q + \varphi)\\ &= \alpha(q) + f_n(\varphi)\alpha^+(q) - f_n(\psi)\alpha^-(q) - f_n(\varphi) f_n(\psi)\alpha^{-+}(q),\end{aligned}$$

but also

$$\begin{aligned}\alpha(q + \varphi - \psi) &= \alpha(q - \psi) + f_n(\varphi)\alpha^+(q - \psi)\\ &= \alpha(q) - f_n(\psi)\alpha^-(q) + f_n(\varphi)\alpha^+(q) - f_n(\varphi) f_n(\psi)\alpha^{+-}(q).\end{aligned}$$

Thus $f_n(\varphi) f_n(\psi)\alpha^{-+}(q) = f_n(\varphi) f_n(\psi)\alpha^{+-}(q)$, and since this holds for any φ, ψ over any extension, by Lemma 0.4 we find $\alpha^{+-} = \alpha^{-+}$.

If we now take $\varphi = \psi$, the above formula gives

$$f_n(\varphi)\alpha^+(q) - f_n(\varphi)\alpha^-(q) = f_n(\varphi) f_n(\varphi)\alpha^{+-}(q),$$

which gives the result, using $f_n(\varphi) f_n(\varphi) = \{-1\}^n f_n(\varphi)$ and again Lemma 0.4. □

In view of this proposition, we may write $\alpha^{r+,s-} \in M(n)$ for any $\alpha \in M(n)$ and $r, s \in \mathbb{N}$, defined as applying r times Φ^+ to α, and s times Φ^-, in any order.

We also call $\Phi = \Phi^+$ the *shifting operator*, as justified by the following elementary result.

Proposition 3.3. *Let $n \in \mathbb{N}^*$. For any $d \in \mathbb{N}$, $\Phi(f_n^{d+1}) = f_n^d$ (and $\Phi(f_n^0) = 0$).*

Proof. We need to show $f_n^{d+1}(q+\varphi) = f_n^{d+1}(q) + f_n(\varphi) \cdot f_n^d(q)$, which is an immediate consequence of Proposition 2.2. □

The action of Φ^- on the f_n^d is more complicated, reflecting the fact that f_n^d behaves very nicely with respect to sums of Pfister forms, but quite poorly for differences of those.

Proposition 3.4. *Let $n, d \in \mathbb{N}^*$. Then*

$$(f_n^d)^- = \sum_{k=0}^{d-1} (-1)^{d-k-1} \{-1\}^{n(d-k-1)} f_n^k.$$

Proof. Let $q \in I^n(K)$ and $\varphi \in \mathrm{Pf}_n(K)$. Then

$$f_n^d(q-\varphi) = \sum_{k=0}^{d} f_n^k(q) f_n^{d-k}(-\varphi) = f_n^d(q) + \sum_{k=0}^{d-1} (-1)^{d-k} \{-1\}^{n(d-k-1)} f_n(\varphi) f_n^k(q)$$

using formula (2.3). □

Apart from its action on the f_n^d, the main property of Φ_n^ε is the following:

Proposition 3.5. *Let $n \in \mathbb{N}^*$ and $\varepsilon = \pm 1$. The morphism Φ_n^ε induces for any $d \in \mathbb{N}$ an exact sequence*

$$0 \to A(k)/A^{\geqslant d+n}(k) \to M(n)/M^{\geqslant d+n}(n) \xrightarrow{\Phi_n^\varepsilon} M(n)/M^{\geqslant d}(n).$$

In particular, the kernel of Φ_n^ε is the submodule of constant invariants in $M(n)$.

Proof. If $\alpha, \beta \in M(n)$ are congruent modulo $M^{\geqslant d+n}(n)$, then since $\Phi^\varepsilon(M^{\geqslant d+n}(n))$ is included in $M^{\geqslant d}(n)$, α^ε and β^ε are congruent modulo $M^{\geqslant d}(n)$.

Let $\alpha \in M(n)$ be such that $\alpha^\varepsilon \in M^{\geqslant d}(n)$. Then for any $q \in I^n(K)$ and any $\varphi \in \mathrm{Pf}_n(K)$, we have $\alpha(q+\varepsilon\varphi) \equiv \alpha(q)$ modulo $A^{\geqslant n+d}(K)$, and also by symmetry $\alpha(q-\varepsilon\varphi) \equiv \alpha(q)$. Since we can always write $q = q_1 - q_2$, where the q_i are sums of n-fold Pfister forms, then by simple induction on the lengths of the sums, $\alpha(q) \equiv \alpha(0)$ modulo $A^{\geqslant n+d}(K)$ (where $\alpha(0)$ is seen as a constant invariant).

Taking a large enough d, and since the filtration on $A(K)$ is separated, we see that $\alpha^\varepsilon = 0$ implies $\alpha = \alpha(0)$. □

Corollary 3.6. *Let $n \in \mathbb{N}^*$ and let $\varepsilon = \pm 1$. If $M'(n)$ is the submodule of $M(n)$ generated by the f_n^d for $d \in \mathbb{N}$, then Φ_n^ε induces an exact sequence of filtered $A(k)$-modules*

$$0 \to A(k) \to M'(n) \xrightarrow{\Phi_n^\varepsilon} M'(n)[-n] \to 0.$$

Proof. The only thing left to check is surjectivity, but this is easily implied by Propositions 3.3 for Φ^+ and 3.4 for Φ^-. □

Remark 3.7. All this implies that Φ may be seen as some kind of differential operator: if we know α^+ for some invariant α, we may "integrate" to find α, with a certain integration constant. Precisely, if $\alpha^+ = \sum a_d f_n^d$, then $\alpha = \alpha(0) + \sum a_d f_n^{d+1}$ (and we show in the next section that such a decomposition always holds). We use this method extensively to compute some invariants α by "induction on shifting".

4. Classification of invariants

The main goal of this article, and this section, is to show that any $\alpha \in M(n)$ can be expressed uniquely as a combination $\sum_d a_d f_n^d$. The next proposition gives the first step:

Proposition 4.1. *Let $n \in \mathbb{N}^*$ and $d \in \mathbb{N}$. The $A(k)/A^{\geqslant d}(k)$-module $M(n)/M^{\geqslant d}(n)$ is generated by the f_n^k with $nk < d$.*

Proof. We use induction on d. For $d = 0$, this is trivial since $M^{\geqslant 0}(n) = M(n)$. Suppose the property holds up to $d-1$, and let $\alpha \in M(n)$; we write $\bar{\alpha} \in M(n)/M^{\geqslant d}(n)$ for its residue class. By induction, $\Phi(\bar{\alpha}) = \sum a_k f_n^k$ with $nk < d-n$, so if we set $\beta = \alpha - \sum a_k f_n^{k+1}$ we get $\Phi(\bar{\beta}) = 0$. From there, β is congruent modulo $M^{\geqslant d}(n)$ to a constant invariant a_{-1}, hence $\bar{\alpha} = \sum a_{k-1} f_n^k$ with $nk < d$. □

The problem is that to express an invariant in terms of the f_n^d, it is in general necessary to use an infinite combination, as the following example illustrates.

Example 4.2. Consider the case $A = W$. Let $\alpha(q) = \langle \mathrm{disc}(q) \rangle$; it is a Witt invariant of I. Then $\alpha^+ = -\alpha$; indeed,

$$\langle \mathrm{disc}(q + \langle\langle a \rangle\rangle) \rangle = \langle \mathrm{disc}(q) a \rangle = \langle \mathrm{disc}(q) \rangle - \langle\langle a \rangle\rangle \langle \mathrm{disc}(q) \rangle.$$

Thus α cannot be written as a finite combination of the f_1^d (since the length of such a combination strictly decreases when applying Φ^+). On the other hand, we may write it (at least formally for now) as

$$\alpha = \sum_{d \in \mathbb{N}} (-1)^d f_1^d.$$

But such an infinite combination may not always be well-defined: since the f_n^d take values in $A^{\geqslant m}$ for increasing values of m, any $\sum_{d \in \mathbb{N}} a_d f_n^d$ is well-defined as an invariant with values in the completion of A with respect to its filtration, but usually not in A itself, as the next example shows.

Example 4.3. If k is formally real, then $\sum_d f_1^d$ sends $-\langle\langle -1 \rangle\rangle$ to $\sum_{d \in \mathbb{N}} (-1)^d \{-1\}^d$, which is not in $A(k)$ (but is in its completion).

It readily appears that the trouble is the bad behaviour of the f_n^d with respect to the opposites of Pfister forms. To get a satisfying description of $M(n)$, we introduce a new "basis", with better balance between sums and differences of Pfister forms, such that any infinite combination does take values in A.

Definition 4.4. Let $n \in \mathbb{N}^*$. For any $d \in \mathbb{N}$, we define $g_n^d \in M^{\geqslant nd}(n)$ by

- $g_n^0 = 1$;
- if $d \in \mathbb{N}^*$ is odd, $(g_n^d)^- = g_n^{d-1}$ and $g_n^d(0) = 0$;
- if $d \in \mathbb{N}^*$ is even, $(g_n^d)^+ = g_n^{d-1}$ and $g_n^d(0) = 0$.

If $A = W$ (resp. $A = H$), we sometimes write γ_n^d (resp. $v_{nd}^{(n)}$) for g_n^d.

Corollary 3.6 ensures that these are well-defined. This definition, which balances Φ^+ and Φ^-, gives a reasonable behaviour under both operators:

Proposition 4.5. *Let $n \in \mathbb{N}^*$ and $d \in \mathbb{N}$. Then*

$$(g_n^{d+2})^{+-} = (g_n^{d+2})^{-+} = g_n^d;$$

$$(g_n^{d+1})^+ = \begin{cases} g_n^d & \text{if } d \text{ is odd,} \\ g_n^d + \{-1\}^n g_n^{d-1} & \text{if } d \text{ is even;} \end{cases}$$

$$(g_n^{d+1})^- = \begin{cases} g_n^d & \text{if } d \text{ is even,} \\ g_n^d - \{-1\}^n g_n^{d-1} & \text{if } d \text{ is odd.} \end{cases}$$

Proof. If d is even, then $(g_n^{d+2})^- = g_n^{d+1}$ and $(g_n^{d+1})^+ = g_n^d$, and if d is odd, $(g_n^{d+2})^+ = g_n^{d+1}$ and $(g_n^{d+1})^- = g_n^d$. In any case the first formula is satisfied.

For the remaining two, we use $(g_n^{d+1})^+ - (g_n^{d+1})^- = \{-1\}^n g_n^{d-1}$ coming from Proposition 3.2. We may conclude, arguing according to the parity of d. ☐

We can now write the precise relation between f_n^d and g_n^d:

Proposition 4.6. *Let $n \in \mathbb{N}^*$. For any $d \in \mathbb{N}^*$,*

$$g_n^d = \sum_{k=\lfloor d/2\rfloor+1}^{d} \binom{\lfloor \frac{d-1}{2} \rfloor}{k - \lfloor \frac{d}{2} \rfloor - 1} \{-1\}^{n(d-k)} f_n^k,$$

$$f_n^d = \sum_{k=1}^{d} (-1)^{d-k} \binom{d - \lfloor \frac{k+1}{2} \rfloor - 1}{\lfloor \frac{k}{2} \rfloor - 1} \{-1\}^{n(d-k)} g_n^k.$$

In particular, $(f_n^i)_{i\leqslant d}$ and $(g_n^i)_{i\leqslant d}$ generate the same submodule of $M(n)$.

Proof. Denote by α_d the invariant defined by the right-hand side of the formula for g_n^d. If $d = 2m$, the formula becomes

$$\alpha_d = \sum_{k=m+1}^{2m} \binom{m-1}{k-m-1} \{-1\}^{n(2m-k)} f_n^k,$$

which gives

$$\alpha_d^+ = \sum_{k=m+1}^{2m} \binom{m-1}{k-m-1} \{-1\}^{n(2m-k)} f_n^{k-1},$$

and if $d = 2m+1$ then we get

$$\alpha_d = \sum_{k=m+1}^{2m+1} \binom{m}{k-m-1} \{-1\}^{n(2m+1-k)} f_n^k.$$

Hence

$$\alpha_d^+ = \sum_{k=m+1}^{2m+1} \binom{m}{k-m-1} \{-1\}^{2m+1-k} f_n^{k-1}.$$

We thus have to check that in both cases we find the correct induction formula for α_{d+1}^+ (coming from Proposition 4.5). If $d = 2m+1$, we have to show $\alpha_{2m+2}^+ = \alpha_{2m+1}$, which is immediate given the above formulas. If $d = 2m$, we have to show $\alpha_{2m+1}^+ = \alpha_{2m} + \{-1\}^n \alpha_{2m-1}$, so we need to compare

$$\sum_{k=m}^{2m} \binom{m}{k-m} \{-1\}^{n(2m-k)} f_n^k$$

and

$$\sum_{k=m+1}^{2m} \binom{m-1}{k-m-1} \{-1\}^{n(2m-k)} f_n^k + \sum_{k=m}^{2m-1} \binom{m-1}{k-m} \{-1\}^{n(2m-k)} f_n^k,$$

which are easily seen as being equal using Pascal's triangle.

The formula for f_n^d can be obtained either in a similar fashion, or by inverting the one for g_n^d. Let β_d be the invariant defined by the right-hand side. Then we show that $\beta_d^+ = \beta_{d-1}$, separating the sums according to the parity of k:

$$\begin{aligned}
\beta_d^+ &= (-1)^d \sum_m \binom{d-m-1}{m-1} \{-1\}^{n(d-2m)} (g_n^{2m})^+ \\
&\qquad + (-1)^{d+1} \sum_m \binom{d-m-2}{m-1} \{-1\}^{n(d-2m-1)} (g_n^{2m+1})^+ \\
&= (-1)^d \sum_m \binom{d-m-1}{m-1} \{-1\}^{n(d-2m)} g_n^{2m-1} \\
&\qquad + (-1)^{d+1} \sum_m \binom{d-m-2}{m-1} \{-1\}^{n(d-2m-1)} (g_n^{2m} + \{-1\}^n g_n^{2m-1})
\end{aligned}$$

$$\begin{aligned} &= (-1)^{d+1}\sum_m \binom{d-m-2}{m-1}\{-1\}^{n(d-2m-1)} g_n^{2m} \\ &\qquad + (-1)^d \sum_m \left(\binom{d-m-1}{m-1} - \binom{d-m-2}{m-1}\right)\{-1\}^{n(d-2m)} g_n^{2m-1} \\ &= (-1)^{d-1}\sum_m \binom{d-1-m-1}{m-1}\{-1\}^{n(d-1-2m)} g_n^{2m} \\ &\qquad + (-1)^{d-1+1}\sum_m \binom{d-m-1}{m-2}\{-1\}^{n(d-2m)} g_n^{2m-1}, \end{aligned}$$

which does give α_{d-1}.

The last statement comes from the fact that the transition matrix from $(f_n^d)_d$ to $(g_n^d)_d$ is triangular unipotent. □

The next proposition gives an important consequence of the balance of g_n^d.

Proposition 4.7. *Let $n \in \mathbb{N}^*$, and let $q \in I^n(K)$ be such that $q = \sum_{i=1}^s \varphi_i - \sum_{i=1}^t \psi_i$, where $\varphi_i, \psi_i \in \mathrm{Pf}_n(K)$. Then for any $d > 2\max(s,t)$, we have $g_n^d(q) = 0$.*

Proof. We may add hyperbolic forms in either sum so that $s = t$. Then we prove the statement by induction on s: if $s = 0$ then $q = 0$, so for $d > 0$ we have indeed $g_n^d(q) = 0$ by construction.

If the result holds up to $s-1$ for some $s \in \mathbb{N}^*$, then write $q' = q - \varphi_s$ and $q'' = q' + \psi_s$. We get

$$\begin{aligned} g_n^d(q) &= g_n^d(q') + f_n(\varphi_s)(g_n^d)^+(q') \\ &= g_n^d(q'') - f_n(\psi_s)(g_n^d)^-(q'') + f_n(\varphi_s)(g_n^d)^+(q'') - f_n(\varphi_s) f_n(\psi_s)(g_n^d)^{+-}(q''). \end{aligned}$$

Now according to Proposition 4.5, $(g_n^d)^-$, $(g_n^d)^+$ and $(g_n^d)^{+-}$ may all be expressed as combinations of some g_n^k with $k \geqslant d-2$, so we may apply the induction hypothesis with q''. □

Corollary 4.8. *If $q \in I(K)$ is the Witt class of an r-dimensional form, then $g_1^d(q) = 0$ for any $d > r$.*

Proof. Writing $r = 2m$, if $q = \langle a_1, b_1, \dots, a_m, b_m\rangle$, then $q = \sum_{i=1}^m \langle\langle -a_i\rangle\rangle - \langle\langle b_i\rangle\rangle$, which allows us to conclude using the previous proposition. □

We may now put it all together to prove the central theorem:

Theorem 4.9. *Let $n \in \mathbb{N}^*$, and let $N(n) = A(k)^{\mathbb{N}}$, which is a filtered $A(k)$-module for the filtration $N^{\geqslant m}(n) = \{(a_d)_{d\in\mathbb{N}} \mid a_d \in A^{\geqslant m-nd}\}$. The following applications are mutually inverse isomorphisms of filtered $A(k)$-modules*:

$$\begin{aligned} F &: N(n) \xrightarrow{\sim} M(n), \quad (a_d)_{d\in\mathbb{N}} \mapsto \textstyle\sum_{d\in\mathbb{N}} a_d g_n^d, \\ G &: M(n) \xrightarrow{\sim} N(n), \quad \alpha \mapsto (\alpha^{[d]}(0))_{d\in\mathbb{N}}, \end{aligned}$$

where $\alpha^{[d]} = \alpha^{m+,m-}$ if $d = 2m$, and $\alpha^{[d]} = \alpha^{(m+1)+,m-}$ if $d = 2m+1$.

Proof. First, the application F is well-defined, since according to Proposition 4.7, for any fixed $q \in I^n(K)$ we have $g_n^d(q) = 0$ for large enough d. Then F and G are clearly module morphisms, and the fact that they respect the filtrations is just a reformulation of the fact that g_n^d takes values in $A^{\geqslant nd}$, and that Φ_n^{ε} has degree $-n$. Let $\alpha = \sum_d a_d g_n^d$. Using Proposition 3.5, we see that for any $r, s \in \mathbb{N}$, we can ignore the terms for d large enough when we compute $\alpha^{r+,s-}(0)$. Thus it is easy to see from Proposition 4.5 that $a_{2m} = \alpha^{m+,m-}(0)$ and $a_{2m+1} = \alpha^{(m+1)+,m-}(0)$, which shows that $G \circ F = \mathrm{Id}$.

We now prove that G is injective, which finishes the proof of the theorem. Let $\alpha \in \mathrm{Ker}(G)$, and let $d \in \mathbb{N}$. According to Proposition 4.1, and using the last statement of Proposition 4.6, we see that α is congruent to some combination $\sum_{nk<d} a_k g_n^k$ modulo $M^{\geqslant d}(n)$. Now the exact sequence in Proposition 3.5 shows that $a_k \equiv \alpha^{[k]}(0)$ modulo $A^{\geqslant d-nk}(k)$, so since $\alpha^{[k]}(0) = 0$, $a_k \in A^{\geqslant d-nk}(k)$. This in turn implies that $\sum_{nk\leqslant d} a_k g_n^k \in M^{\geqslant d}(n)$, and thus $\alpha \in M^{\geqslant d}(n)$. Since this is true for any $d \in \mathbb{N}$, we may conclude that $\alpha = 0$. □

Corollary 4.10. *Let $n \in \mathbb{N}^*$ and let $\varepsilon = \pm 1$. There is an exact sequence of filtered $A(k)$-modules*

$$0 \to A(k) \to M(n) \xrightarrow{\Phi_n^{\varepsilon}} M(n)[-n] \to 0.$$

Proof. Like for Corollary 3.6, the only thing left to prove after Proposition 3.5 is the surjectivity of Φ_n^{ε}, but it is an easy consequence of Theorem 4.9. □

Corollary 4.11. *Let $n \in \mathbb{N}^*$ and $\alpha \in M(n)$. There is a unique sequence $(a_d)_{d\in\mathbb{N}}$ with $a_d \in A(k)$ such that for any $q \in I^n(K)$, the infinite sum $\sum_{d\in\mathbb{N}} a_d f_n^d(q)$ exists in $A(K)$ and is equal to $\alpha(q)$. Furthermore, for all $d \in \mathbb{N}$, we have $a_d = \alpha^{d+}(0)$.*

Proof. If such a sequence exists, then using Proposition 3.5 we find that $\alpha^{i+}(0) \equiv a_i$ modulo $A^{\geqslant dn}(k)$ for all $i \leqslant d$, so for a fixed i we can make d go to infinity, and we find that indeed $a_d = \alpha^{d+}(0)$, which shows uniqueness.

For existence, write $\alpha = \sum_d b_d g_n^d$, and decompose each g_n^d in terms of the f_n^i using Proposition 4.6. Then we find a decomposition of α in terms of f_n^d which is valid pointwise, and the a_d we find are well-defined in $A(k)$ since each a_d is a combination of a finite number of b_i (using that f_n^i appears in the decomposition of g_n^d only if $d \leqslant 2i$). □

Remark 4.12. In particular, any invariant of I^n with values in $H^d(-, \mu_2)$ may be lifted to an invariant with values in I^d.

Remark 4.13. If k is not a formally real field, then for large enough d we have $\{-1\}^d = 0$, and thus according to formula (2.3), $f_n^d(-\varphi) = 0$ for any $\varphi \in \mathrm{Pf}_n(K)$. This implies that in this case, for any $q \in I^n(K)$ we have $f_n^d(q) = 0$ for large enough d (for the same reasons as in Corollary 2.4), and so we may use the f_n^d instead of the g_n^d in the theorem (with $G(\alpha) = (\alpha^{d+}(0))_d)$. In the extreme case where -1 is a square in k, we actually even get $f_n^d = g_n^d$, as can be seen from Proposition 4.6. On the other hand, Example 4.3 shows that we cannot use the f_n^d if k is formally real. What happens in this case is that an arbitrary infinite combination of the f_n^d does correspond to a combination of the g_n^d (using Proposition 4.6), but with coefficients in the completion of A with respect to its filtration.

Remark 4.14. We may construct cohomological invariants α such that, even though the degree of $\alpha(q)$ is bounded for fixed q, it is unbounded when q varies (for instance, $\alpha = \sum_d g_n^d$). This reflects in some sense the "infinite" nature of I^n, and it is a behaviour that does not appear for invariants of algebraic groups. The submodule $M'(n)$ of uniformly bounded cohomological invariant is the submodule generated by the f_n^d (or by the g_n^d). We may write that $M(n) = \mathrm{Inv}(I^n, \varinjlim H^{\leqslant d}(-, \mu_2))$, while $M'(n) = \varinjlim \mathrm{Inv}(I^n, H^{\leqslant d}(-, \mu_2))$.

5. Algebra structure

Since $M(n)$ is not only an $A(k)$-module, but also an algebra, we wish to understand how the product can be expressed in terms of the basic elements f_n^d.

For this section, if $d, p, q \in \mathbb{N}$ are such that $p+q \leqslant d$, we set

$$C_{p,q}^d = \frac{d!}{p! \cdot q! \cdot (d-p-q)!}.$$

This is just a more compact notation for the usual multinomial

$$\binom{d}{p,\, q,\, d-p-q}.$$

Proposition 5.1. *Let $n \in \mathbb{N}^*$, and $\varepsilon = \pm 1$. Then for any $\alpha, \beta \in M(n)$,*

$$\Phi^\varepsilon(\alpha\beta) = \Phi^\varepsilon(\alpha)\beta + \alpha\Phi^\varepsilon(\beta) + \varepsilon\{-1\}^n \Phi^\varepsilon(\alpha)\Phi^\varepsilon(\beta).$$

Proof. Let $q \in I^n(K)$ and $\varphi \in \mathrm{Pf}_n(K)$. Then

$$\begin{aligned}(\alpha\beta)(q+\varepsilon\varphi) &= (\alpha(q)+\varepsilon f_n(\varphi)\alpha^\varepsilon(q))\cdot(\beta(q)+\varepsilon f_n(\varphi)\beta^\varepsilon(q)) \\ &= (\alpha\beta)(q)+\varepsilon f_n(\varphi)\big((\alpha^\varepsilon\beta)(q)+(\alpha\beta^\varepsilon)(q)+\varepsilon\{-1\}^n(\alpha^\varepsilon\beta^\varepsilon)(q)\big). \quad \square\end{aligned}$$

Proposition 5.2. *Let $n \in \mathbb{N}^*$ and $s, t \in \mathbb{N}$. Then*

$$f_n^s \cdot f_n^t = \sum_{d=\max(s,t)}^{s+t} C_{d-s,d-t}^d \{-1\}^{n(s+t-d)} f_n^d.$$

Proof. First note that both sides of the equality have the same value in 0 (which is 1 if $s = t = 0$ and 0 otherwise). So we just need to show that applying Φ to both sides of the equation gives the same expression.

Now Proposition 5.1 gives

$$\Phi(f_n^s \cdot f_n^t) = f_n^s \cdot f_n^{t-1} + f_n^{s-1} \cdot f_n^t + \{-1\}^n f_n^{s-1} \cdot f_n^{t-1}. \tag{5.3}$$

We proceed by induction, say on (s, t) with lexicographical order. First the result is clear if $s = 0$ or $t = 0$. Then by induction we can replace each term in (5.3) and rearrange them to find, for $\Phi(f_n^s \cdot f_n^t)$,

$$\binom{s}{t}\{-1\}^{nt} f_n^{s-1} + \binom{s+t}{t} f_n^{s+t-1} + \sum_{d=s}^{s+t-2} C_{d-s+1,d-t+1}^{d+1}\{-1\}^{n(s+t-d-1)} f_n^d, \tag{5.4}$$

where for the coefficient before f_n^{s-1} we use $\binom{s-1}{t} + \binom{s-1}{t-1} = \binom{s}{t}$, for that of f_n^{s+t-1} we use $\binom{s+t-1}{t} + \binom{s+t-1}{t-1} = \binom{s+t}{t}$, and for the other terms we use

$$C_{d-s+1,d-t}^d + C_{d-s,d-t+1}^d + C_{d-s+1,d-t+1}^d = C_{d-s+1,d-t+1}^{d+1}.$$

We can then compute that applying Φ to the right-hand side of the equality in the statement of the proposition yields exactly (5.4). □

Of course there is a corresponding formula for the products of the g_n^d, but it turns out that it is much more involved, and we do not address it here. This means that although we have a nice module isomorphism between $M(n)$ and $A(k)^{\mathbb{N}}$, transporting the algebra structure of $M(n)$ to $A(k)^{\mathbb{N}}$ is not as convenient. On the other hand, if we use the f_n^d we only have a module isomorphism between $M(n)$ and a submodule of $A(k)^{\mathbb{N}}$, which is hard to describe, but we can transport the product in a reasonably easy way.

There are several cases where the formula of Proposition 5.2 can be greatly simplified by studying the parity of the multinomials that appear. We introduce some notation: if $s, t \in \mathbb{N}$, we write $s \vee t$ (resp. $s \wedge t$) for the integer obtained by applying a bitwise *or* (resp. a bitwise *and*) to the binary representations of s and t. In particular, $s \vee t + s \wedge t = s + t$.

Lemma 5.5. *Let $d, s, t \in \mathbb{N}$ be such that $\max(s, t) \leqslant d \leqslant s + t$. Then $C_{d-s,d-t}^d$ is odd if and only if $d = s \vee t$.*

Proof. It is well-known that for any $a \in \mathbb{N}$, the 2-adic valuation of $a!$ is $a - f(a)$, where $f(a)$ is the number of 1's in the binary representation of a. Then

$$\begin{aligned} &v_2(C_{d-s,d-t}^d) \\ &\quad = (d-f(d))-(s+t-d-f(s+t-d))-(d-s-f(d-s))-(d-t-f(d-t)) \\ &\quad = f(s+t-d)+f(d-s)+f(d-t)-f(d). \end{aligned}$$

But it is easily seen that for any $a, b \in \mathbb{N}$, $f(a+b) \leqslant f(a) + f(b)$, with equality if and only if $a \wedge b = 0$. Thus $C_{d-s,d-t}^d$ is odd if and only if $s + t - d$, $d - s$ and $d - t$ have pairwise disjoint binary representations.

We claim this is equivalent to $d = s \vee t$. Indeed, if $d = s \vee t$ it is obvious, and if $d \neq s \vee t$, consider the weakest bit where d and $s \vee t$ differ; there are several possibilities for the bits of s, t and d in this slot: s has 1 and d has 0, t has 1 and d has 0, or s and t have 0 and d has 1. In all these cases, at least two numbers among $d - s$, $d - t$ and $s + t - d$ have a 1 in this slot, and their binary representations are thus not disjoint. □

Then we can state the following:

Corollary 5.6. *Let $n \in \mathbb{N}^*$ and $s, t \in \mathbb{N}$. If $A = H$, then*

$$u_{ns}^{(n)} \cup u_{nt}^{(n)} = (-1)^{n(s\wedge t)} \cup u_{n(s\vee t)}^{(n)}.$$

Proof. Since $H^*(k, \mu_2)$ is a ring of characteristic 2, by Lemma 5.5 the only potentially nonzero term in the formula of Proposition 5.2 is $\{-1\}^{s\wedge t} f_n^{s\vee t}$. □

Remark 5.7. This is very reminiscent of the formula for the product of Stiefel–Whitney classes, since $w_s \cup w_t = (-1)^{s\wedge t} \cup w_{s\vee t}$. When -1 is a square, this is easily explained by the fact that $u_d^{(1)}$ coincides with the Stiefel–Whitney map w_d (see Remark 9.10), but in general w_d is not well-defined on Witt classes so the formulas are really different phenomena.

Corollary 5.8. *Let $n \in \mathbb{N}^*$ and $s, t \in \mathbb{N}$. If -1 is a square in k, then $f_n^s \cdot f_n^t$ equals f_n^{s+t} if $s \wedge t = 0$, and 0 otherwise.*

Proof. Note that in this situation $A(k)$ is also a ring of characteristic 2, so the same reasoning as in Corollary 5.6 applies, but this time if $s \wedge t \neq 0$ the term is also 0. □

Remark 5.9. Consider the case $A = H$, and the submodule $M'(n) \subset M(n)$ generated by the $u_{nd}^{(n)}$, which is the subalgebra of cohomological invariants with uniformly bounded degree. Then from Corollary 5.6 we find a very simple algebra presentation of $M'(n)$: the (commuting) generators are $x_i = u_{n2^i}^{(n)}$, and the relations are given by $x_i^2 = \{-1\}^{n2^i} x_i$.

6. Restriction from I^n to I^{n+1}

For any $m, n \in \mathbb{N}^*$ with $m \geqslant n$, there is an obvious restriction morphism

$$\rho_{n,m} : M(n) \to M(m), \quad \alpha \mapsto \alpha_{|I^m}. \tag{6.1}$$

Given the definition of f_n^d, if we want to express $(f_n^d)_{|I^{n+1}}$ in terms of the f_{n+1}^k, it is natural to try to express π_n^d in terms of the π_{n+1}^k in $\mathrm{GW}(K)$.

Proposition 6.2. *Let $n \in \mathbb{N}^*$. For any $d \in \mathbb{N}^*$, we have*

$$\pi_n^d = \sum_{d/2 \leqslant k \leqslant d} \binom{k}{d-k} 2^{(d-k)(n-1)} \pi_{n+1}^k.$$

Proof. We define $p_n(t) = t + 2^{n-1}t^2 \in \mathbb{Z}[t]$. Then recall that $(\pi_n)_t = \lambda_{h_n(t)}$, where $h_n = x_n^{\circ -1}$, and x_n is defined recursively by $x_{n+1} = p_n \circ x_n$. Thus we have the formula $h_n = h_{n+1} \circ p_n$, and

$$(\pi_n)_t = (\pi_{n+1})_{p_n(t)}.$$

Therefore we find

$$\sum_d \pi_n^d \cdot t^d = \sum_k \pi_{n+1}^k (t + 2^{n-1}t^2)^k = \sum_k \sum_{k \leqslant d \leqslant 2k} \binom{k}{d-k} 2^{(d-k)(n-1)} \pi_{n+1}^k \cdot t^d,$$

which gives the result. □

Then we deduce the corresponding results for our invariants.

Corollary 6.3. *Let $n, d \in \mathbb{N}^*$. If $A = W$ then*

$$(\bar{\pi}_n^d)_{|I^{n+1}} = \sum_{d/2 \leqslant k \leqslant d} \binom{k}{d-k} \langle\langle -1 \rangle\rangle^{(d-k)(n-1)} \bar{\pi}_{n+1}^k.$$

Proof. This is an immediate consequence of the proposition, given that in $W(K)$ we have $\langle\langle -1 \rangle\rangle = 2$. □

Corollary 6.4. *Let $n, d \in \mathbb{N}^*$. If $A = H$ then*

$$(u_{nd}^{(n)})_{|I^{n+1}} = \begin{cases} (-1)^{m(n-1)} \cup u_{(n+1)m}^{(n+1)} & \text{if } d = 2m, \\ 0 & \text{if } d \text{ is odd.} \end{cases}$$

Proof. This is also a consequence of the proposition, but we have to notice that when we apply e_{nd} to the formula, the terms corresponding to $k > d/2$ vanish. Indeed, in this case $\langle\langle -1 \rangle\rangle^{(d-k)(n-1)} \pi_{n+1}^k$ sends $\hat{I}^{n+1}(K)$ to $\hat{I}^r(K)$ with

$$r = (d-k)(n-1) + k(n+1) = d(n-1) + 2k > nd.$$

Thus, composing with e_{nd} gives zero.

So only the term $k = d/2$ remains (and only when d is even). □

Remark 6.5. In particular, for cohomological invariants, and when $n = 1$, we get the simple formula $(u_{2d}^{(1)})_{|I^2} = u_{2d}^{(2)}$, which shows that any cohomological invariant of I^2 extends (not uniquely) to I. On the other hand, for $n \geqslant 3$ and $d \geqslant 1$, $u_{nd}^{(n)}$ never extends to I^{n-1}. This vastly generalizes the familiar facts that e_2 extends to I, but e_3 does not extend to I^2.

Remark 6.6. Suppose -1 is a square in k, and take $n \geqslant 2$. Then in the case of Witt invariants, $\bar{\pi}_n^d$ is independent of n, and in the case of cohomological invariants the restriction of any $\alpha \in M(n)$ to I^{n+1} is constant.

As an application of Corollary 6.4, we may improve a result of Kahn [2005]: he shows in the proof of Proposition 3.3 that if $H^r(K, \mu_2)$ has symbol length at most $l \in \mathbb{N}$, then any element of $H^{r(l+1)}(K, \mu_2)$ is a multiple of $(-1) \in H^1(K, \mu_2)$. We would like to thank Karim Becher for fruitful discussions about this application during a visit in Antwerp.

Proposition 6.7. *Let $r \in \mathbb{N}^*$, and assume that $H^r(K, \mu_2)$ has symbol length at most $l \in \mathbb{N}$. Then for any $d > l$, we have*

$$H^{rd}(K, \mu_2) \subset (-1)^{(r-1)\lceil (d-l)/2 \rceil} \cup H^*(K, \mu_2).$$

In particular, any element of $H^m(K, \mu_2)$ for $m \geqslant r(l+1)$ is a multiple of

$$(-1)^{r-1} \in H^{r-1}(K, \mu_2).$$

Proof. It is enough to prove the result for Galois symbols: let $\alpha \in H^{rd}(K, \mu_2)$ be a symbol, and write $\alpha = \alpha_1 \cup \cdots \cup \alpha_d$ with $\alpha_i \in H^r(K, \mu_2)$. Then we set $\varphi_i \in \mathrm{Pf}_r(K)$ such that $e_r(\varphi_i) = \alpha_i$, and $q = \sum_i \varphi_i \in I^r(K)$. According to Corollary 2.4, we have $\alpha = u_{rd}^{(r)}(q)$.

Now by hypothesis, $q = q' + x$, where $q' \in I^r(K)$ can be written as a sum of l or less r-fold Pfister forms, and $x \in I^{r+1}(K)$. We have

$$\alpha = u_{rd}^{(r)}(q' + x) = \sum_{k=0}^{rd} u_{rk}^{(r)}(q') \cup u_{r(d-k)}^{(r)}(x). \tag{6.8}$$

But Corollary 2.4 shows that $u_{rk}^{(r)}(q') = 0$ when $k > l$, and Corollary 6.4 shows that $u_{r(d-k)}^{(r)}(x)$ is a multiple of $(-1)^{(r-1)\lceil (d-k)/2 \rceil}$. It thus follows from (6.8) that α is a multiple of $(-1)^{(r-1)\lceil (d-l)/2 \rceil}$. □

7. Similitudes

In this section we study the behaviour of invariants with respect to similitudes.

Proposition 7.1. *There is a unique morphism of filtered $A(k)$-modules*

$$\Psi : \mathrm{Inv}(W, A) \to \mathrm{Inv}_0(W, A)[-1], \quad \alpha \mapsto \widetilde{\alpha}$$

such that

$$\alpha(\langle \lambda \rangle q) = \alpha(q) + \{\lambda\} \widetilde{\alpha}(q) \tag{7.2}$$

for any $\alpha \in \mathrm{Inv}(W, A)$, $q \in F(K)$ and $\lambda \in K^*$.

If F is a subfunctor of W such that $F(L)$ is stable under similitudes for any L/k, and $0 \in F(k)$, then Ψ restricts to a morphism $\mathrm{Inv}(F, A) \to \mathrm{Inv}_0(F, A)[-1]$. In particular, for any $n \in \mathbb{N}^*$ we get a filtered morphism $M(n) \to M_0(n)[-1]$.

Proof. Let $\alpha \in \mathrm{Inv}(F, A^{\geqslant d})$ for some $d \in \mathbb{N}$ and $q \in F(K)$. For any $\lambda \in L^*$, where L/K is any field extension, we set $\beta_q(\lambda) = \alpha(\langle \lambda \rangle q)$.

Then β_q is an invariant over K of square classes, with values in A. Now the functor of square classes is isomorphic to Pf_1, so we may apply Lemma 0.5: there are uniquely determined $x_q, y_q \in A(K)$ such that $\beta_q(\lambda) = x_q + \{\lambda\} \cdot y_q$ for all λ. Taking $\lambda = 1$ we see that $x_q = \alpha(q)$, and we set $\widetilde{\alpha}(q) = y_q$.

The uniqueness of y_q implies that $\widetilde{\alpha} \in \mathrm{Inv}(F, A)$. The fact that $\{\lambda\} \cdot y_q \in A^{\geqslant d}(L)$ for all $\lambda \in L^*$ shows according to Lemma 0.4 that $y_q \in A^{\geqslant d-1}(K)$, so as a filtered morphism Ψ has degree -1. It is clear that if $q = 0$, then $\alpha(\langle\lambda\rangle q) = \alpha(q) + \{\lambda\} \cdot 0$, so $\widetilde{\alpha}(0) = 0$, which means that $\widetilde{\alpha}$ is normalized. □

We first establish some basic properties of Ψ:

Proposition 7.3. *Let* $\alpha, \beta \in \mathrm{Inv}(W, A)$. *Then*

$$\Psi(\alpha\beta) = \Psi(\alpha)\beta + \alpha\Psi(\beta) + \{-1\}\Psi(\alpha)\Psi(\beta).$$

Proof. Let $q \in W(K)$ and $\lambda \in K^*$. Then

$$\begin{aligned}(\alpha\beta)(\langle\lambda\rangle q) &= (\alpha(q) + \{\lambda\}\widetilde{\alpha}(q))(\beta(q) + \{\lambda\}\widetilde{\beta}(q)) \\ &= (\alpha\beta)(q) + \{\lambda\}\big((\widetilde{\alpha}\beta)(q) + (\alpha\widetilde{\beta})(q) + \{-1\}(\widetilde{\alpha}\widetilde{\beta})(q)\big).\end{aligned}$$ □

Proposition 7.4. *We have* $\Psi^2 = -\delta(A)\Psi$.

Proof. For any extension L/K and any $\lambda, \mu \in L^*$,

$$\begin{aligned}\alpha(\langle\lambda\mu\rangle q) &= \alpha(\langle\lambda\rangle q) + \{\mu\}\widetilde{\alpha}(\langle\lambda\rangle q) \\ &= \alpha(q) + \{\lambda\}\widetilde{\alpha}(q) + \{\mu\}\widetilde{\alpha}(q) + \{\lambda, \mu\}\widetilde{\widetilde{\alpha}}(q) \\ &= \alpha(q) + \{\lambda\mu\}\widetilde{\alpha}(q) + \{\lambda, \mu\}(\delta\widetilde{\alpha}(q) + \widetilde{\widetilde{\alpha}}(q)),\end{aligned}$$

using formula (0.2) for the last equality. We also have

$$\alpha(\langle\lambda\mu\rangle q) = \alpha(q) + \{\lambda\mu\}\widetilde{\alpha}(q),$$

so $\{\lambda, \mu\}(\delta\widetilde{\alpha}(q) + \widetilde{\widetilde{\alpha}}(q)) = 0$. Since this holds for any λ, μ over any extension, we may conclude that $\widetilde{\widetilde{\alpha}}(q) = -\delta\widetilde{\alpha}(q)$. □

Remark 7.5. By definition, $\widetilde{\alpha} = 0$ if and only if $\alpha(\langle\lambda\rangle q) = \alpha(q)$, that is to say, α is *invariant under similitudes*. But the previous proposition suggests that in the case $A = W$, $\widetilde{\alpha} = -\alpha$ should also be an interesting property (notably, it is always satisfied by invariants of the form $\widetilde{\beta}$). And indeed, it is easily seen to be equivalent to $\alpha(\langle\lambda\rangle q) = \langle\lambda\rangle\alpha(q)$, in which case we say α is *compatible with similitudes*. Then the proposition shows that any α may be uniquely decomposed as a sum $\alpha = \beta + \gamma$ with β compatible with similitudes, and γ invariant under similitudes. Precisely, $\beta = -\widetilde{\alpha}$ and $\gamma = \alpha + \widetilde{\alpha}$.

From a less intrinsic point of view, if α is a finite combination of the f_n^d, then by definition of the f_n^d it can be seen as a composition

$$I^n(K) \xrightarrow{\sim} \hat{I}^n(K) \subset \mathrm{GW}(K) \xrightarrow{h} \mathrm{GW}(K) \to W(K),$$

where h is a combination of the λ^i. Then β corresponds to selecting only the odd i, while γ corresponds to the even terms. Thus it makes sense to call β the *odd part* of α, and γ its *even part*. This decomposition has no clear equivalent for cohomological invariants.

We now want to describe the action of Ψ on our basic invariants. It turns out that it is much easier to deal with the g_n^d than the f_n^d in this situation.

Proposition 7.6. *Let $n, d \in \mathbb{N}^*$. Then*

$$\widetilde{g_n^d} = \begin{cases} -\delta(A) g_n^d & \text{if } d \text{ is odd,} \\ \{-1\}^{n-1} g_n^{d-1} & \text{if } d \text{ is even.} \end{cases}$$

Proof. We prove the proposition by induction on d. If $d = 1$, the statement means that

$$f_n(\langle\lambda\rangle q) = f_n(q) - \delta\{\lambda\} f_n(q),$$

which is true whether $A = W$ or $A = H$.

Now suppose the proposition holds until $d-1$, for some $d \geqslant 2$. Since $\widetilde{g_n^d}$ is normalized, it is enough to compute $\widetilde{g_n^d}^+$. Let L/K be any extension, and take $q \in I^n(K)$, $\varphi \in \mathrm{Pf}_n(L)$ and $\lambda \in L^*$. Then

$$\begin{aligned} g_n^d(\langle\lambda\rangle(q+\varphi)) &= g_n^d(q+\varphi) + \{\lambda\}\widetilde{g_n^d}(q+\varphi) \\ &= g_n^d(q) + f_n(\varphi)(g_n^d)^+(q) + \{\lambda\}\widetilde{g_n^d}(q) + \{\lambda\} f_n(\varphi)\widetilde{g_n^d}^+(q), \end{aligned}$$

so if we consider generic λ and φ and take residues, we find exactly $\widetilde{g_n^d}^+(q)$.

On the other hand, if we write $\varphi = \langle\langle a\rangle\rangle\psi$, we can compute

$$g_n^d(\langle\lambda\rangle(q+\varphi)) = g_n^d(\langle\lambda\rangle q + \langle\langle\lambda a\rangle\rangle\psi - \langle\langle\lambda\rangle\rangle\psi)$$

using successively on each term Φ^+ relative to $\langle\langle\lambda a\rangle\rangle$, Φ^- relative to $\langle\langle\lambda\rangle\rangle$ and Ψ relative to $\langle\lambda\rangle$, to get an 8-term sum. Again considering generic λ, a and ψ, taking residues, and comparing to the previous computation, we find

$$\widetilde{g_n^d}^+ = -\delta(g_n^d)^+ - \widetilde{(g_n^d)^+} + \{-1\}^{n-1}(g_n^d)^{+-} + \{-1\}^n \widetilde{(g_n^d)^{+-}}, \tag{7.7}$$

using equations (0.2) and (0.3) several times.

If d is even, then $(g_n^d)^+ = g_n^{d-1}$ and $(g_n^d)^{+-} = g_n^{d-2}$, so by induction

$$\widetilde{(g_n^d)^+} = -\delta g_n^{d-1} \quad \text{and} \quad \widetilde{(g_n^d)^{+-}} = \{-1\}^{n-1} g_n^{d-3}.$$

Thus, from (7.7), we get

$$\widetilde{g_n^d}^+ = -\delta g_n^{d-1} + \delta g_n^{d-1} + \{-1\}^{n-1}(g_n^{d-2} + \{-1\}^n g_n^{d-3}) = \{-1\}^{n-1}(g_n^{d-1})^+,$$

which is the expected formula (we need to be a little careful with the case $d = 2$, but we can check that the reasoning still holds if we say that $g_n^{-1} = 0$).

Similarly, if d is odd, $(g_n^d)^+ = g_n^{d-1} + \{-1\}^n g_n^{d-2}$ and $(g_n^d)^{+-} = g_n^{d-2}$, so

$$\widetilde{(g_n^d)^+} = \{-1\}^{n-1} g_n^{d-2} - \delta\{-1\}^n g_n^{d-2} = -\{-1\}^{n-1} g_n^{d-2} \quad \text{and} \quad \widetilde{(g_n^d)^{+-}} = -\delta g_n^{d-2}.$$

Then from (7.7),

$$\widetilde{g_n^d}^+ = -\delta(g_n^d)^+ + \{-1\}^{n-1} g_n^{d-2} + \{-1\}^{n-1} g_n^{d-2} - \delta\{-1\}^n g_n^{d-2} = -\delta(g_n^d)^+$$

using (0.3), which gives the conclusion. □

Corollary 7.8. *The module* $\mathrm{Inv}(I^n/\sim, A)$ *of invariants of similarity classes of elements in* I^n *is given by the combinations* $\sum_{d\in\mathbb{N}} a_d g_n^d$ *with* $\{-1\}^{n-1} a_{2i+2} = \delta(A) a_{2i+1}$ *for all* $i \in \mathbb{N}$.

Proof. The module $\mathrm{Inv}(I^n/\sim, A)$ is naturally isomorphic to the kernel of Ψ, and if $\alpha = \sum_{d\in\mathbb{N}} a_d g_n^d$, we get

$$\widetilde{\alpha} = \sum_{i\in\mathbb{N}} (\{-1\}^{n-1} a_{2i+2} - \delta(A) a_{2i+1}) g_n^{2i+1},$$

which gives the result. □

The formula for $\widetilde{f_n^d}$ is not particularly enlightening (see Remark 7.10), but we may at least give the values of f_n^d on general Pfister forms (which amounts to computing the values of $\widetilde{f_n^d}$ on Pfister forms). This may be deduced from the previous proposition using Proposition 4.6, but we can give a direct proof.

Proposition 7.9. *Let* $n \in \mathbb{N}^*$ *and* $d \geqslant 2$. *Then for any* $\varphi \in \mathrm{Pf}_n(K)$ *and* $\lambda \in K^*$ *we have*

$$f_n^d(\langle\lambda\rangle\varphi) = (-1)^d \{-1\}^{n(d-1)-1} \{\lambda\} f_n(\varphi).$$

Proof. Write $\varphi = \langle\langle x\rangle\rangle\psi$. Then since $\langle\lambda\rangle\langle\langle x\rangle\rangle = \langle\langle \lambda x\rangle\rangle - \langle\langle\lambda\rangle\rangle$, using (2.3), we get

$$\begin{aligned}
f_n^d(\langle\lambda\rangle\varphi) &= f_n^d(\langle\langle\lambda x\rangle\rangle\psi - \langle\langle\lambda\rangle\rangle\psi) \\
&= f_n^d(-\langle\langle\lambda\rangle\rangle\psi) + \{\lambda x\} f_n^{d-1}(-\langle\langle\lambda\rangle\rangle\psi) \\
&= (-1)^d \{-1\}^{n(d-1)} \{\lambda\} f_{n-1}(\psi) \\
&\qquad + \{\lambda x\} f_{n-1}(\psi) (-1)^{n(d-1)} \{-1\}^{n(d-2)} \{\lambda\} f_{n-1}(\psi) \\
&= (-1)^d \{-1\}^{n(d-1)-1} (\{-1\}\{\lambda\} - \{\lambda x\}\{\lambda\}) f_{n-1}(\psi) \\
&= (-1)^d \{-1\}^{n(d-1)-1} \{\lambda\}\{x\} f_{n-1}(\psi).
\end{aligned}$$

□

Remark 7.10. We can give the general formula for $\widetilde{f_n^d}$ for the record, though we do not prove it:

$$\widetilde{f_n^d} = (-1)^d \sum_{k=1}^{d-1} \binom{d-1}{k-1} \{-1\}^{n(d-k)-1} f_n^k + \begin{cases} 0 & \text{if } d \text{ even,} \\ -\delta(A) f_n^d & \text{if } d \text{ odd.} \end{cases}$$

We can check that if we evaluate this on a Pfister form we retrieve Proposition 7.9, and as an even more special case, formula (2.3).

8. Ramification of invariants

In this short section we establish the behaviour of invariants with respect to residues of discrete valuations (which incidentally was one of the main initial motivations of this article). Let thus (K, v) be a valued field, where v is a rank 1 discrete k-valuation, with valuation ring $\mathcal{O}_K$ and residue field κ (in particular, κ is an extension of k, so it has characteristic not 2).

Recall from [Elman et al. 2008, Lemma 19.10] the so-called second residue map $\partial_\pi : W(K) \to W(\kappa)$, which depends on the choice of a uniformizing element $\pi \in K$. We say that $q \in W(K)$ is *unramified* if $\partial_\pi(q) = 0$, which is independent of the choice of π. Then q is unramified if and only if it has a diagonalization $\langle a_1, \dots, a_r\rangle$ with $a_i \in \mathcal{O}_K^*$.

Recall also from [Garibaldi et al. 2003, §7.9, p. 18] the canonical residue map $\partial : H^d(K, \mu_2) \to H^{d-1}(\kappa, \mu_2)$, which extends to $\partial : H^*(K, \mu_2) \to H^*(\kappa, \mu_2)$. We say that $x \in H^*(K, \mu_2)$ is unramified if $\partial(x) = 0$.

Moreover, from [Elman et al. 2008, Lemma 19.14], we have $\partial_\pi(I^d(K)) \subset I^{d-1}(\kappa)$, and using for instance [Elman et al. 2008, Proposition 101.8] we get for any $d \in \mathbb{N}^*$ a commutative diagram

$$\begin{array}{ccc} I^d(K) & \xrightarrow{\partial_\pi} & I^{d-1}(\kappa) \\ \big\downarrow{\scriptstyle e_n} & & \big\downarrow{\scriptstyle e_{d-1}} \\ H^d(K, \mu_2) & \xrightarrow{\partial} & H^{d-1}(\kappa, \mu_2). \end{array}$$

Proposition 8.1. *Let $n \in \mathbb{N}^*$ and $q \in I^n(K)$, where K is endowed with a rank 1 discrete k-valuation. If q is unramified, then $\alpha(q) \in A(K)$ is unramified for any $\alpha \in M(n)$.*

Proof. By hypothesis, $\hat{q} \in \hat{I}^n(K)$ comes from an element of $\mathrm{GW}(\mathcal{O}_K)$, so any $\lambda^i(\hat{q})$ also comes from $\mathrm{GW}(\mathcal{O}_K)$, and is unramified. Since π_n^d is a combination of the λ^i with integer coefficients, $\pi_n^d(\hat{q}) \in \hat{I}^{nd}(K)$ is unramified.

Now, tautologically if $A = W$, and applying the above commutative diagram if $A = H$, this implies that $f_n^d(q) \in A^{\geqslant nd}(K)$ is unramified.

Since any $\alpha \in M(n)$ is a combination of the f_n^d with coefficients in $A(k)$, and v is a k-valuation, we can conclude that $\alpha(q) \in A(K)$ is unramified. □

9. Invariants of Quad_{2r}

In [Garibaldi et al. 2003], Serre gives a complete description of $\mathrm{Inv}(\mathrm{Quad}_m, A)$: it is a free $A(k)$-module of rank $n + 1$, with basis $(\lambda^d)_{0 \leqslant d \leqslant m}$ for $A = W$, and the Stiefel–Whitney classes $(w_d)_{0 \leqslant d \leqslant m}$ for $A = H$ (see [Garibaldi et al. 2003, Theorem 27.16 and §17.1]). Clearly any invariant of I restricts to an invariant of Quad_m for any even m, and we want to express it in terms of the given basis.

For practical purposes it is more convenient to introduce a different basis for $\mathrm{Inv}(\mathrm{Quad}_m, W)$ which is the equivalent of the Stiefel–Whitney classes for Witt invariants. We use the notations and definitions from Section 1. Recall from [Elman et al. 2008, §5] that the total Stiefel–Whitney map $w_t : \mathrm{GW}(K) \to \Lambda(H^*(K, \mu_2))$ is the only group morphism such that $w_t(\langle a\rangle) = 1 + (a)t$ for all $a \in K^*$. We generalize this construction:

Proposition 9.1. *There is a unique group morphism*

$$h_t : \mathrm{GW}(K) \to \Lambda(A(K)), \quad x \mapsto h_t(x) = \sum_{d\in\mathbb{N}} h^d(x)t^d$$

such that $h_t(\langle a\rangle) = 1 + \{a\}t$ for all $a \in K^*$. The map h^d takes values in $A^{\geqslant d}(K)$. For any $m \in \mathbb{N}^*$, we write $h^d_m : \mathrm{Quad}_m(K) \to A(K)$ for the restriction of h^d to forms of dimension m. Then $h^d_m \in \mathrm{Inv}(\mathrm{Quad}_m, A^{\geqslant d})$.

If $A = H$, then h^d is the Stiefel–Whitney map w_d. If $A = W$, we write $P^d = h^d$ and $P^d_m = h^d_m$; then for any $q \in \mathrm{Quad}_m(K)$,

$$P^d_m(q) = \sum_{k=0}^{d} (-1)^k \binom{m-k}{d-k} \lambda^k(q). \tag{9.2}$$

In both cases, $(h^d_m)_{0\leqslant d\leqslant m}$ is a basis of the $A(k)$-module $\mathrm{Inv}(\mathrm{Quad}_m, A)$.

Proof. The uniqueness of h_t is obvious since $\mathrm{GW}(K)$ is generated by the $\langle a\rangle$ as an additive group. For $A = H$, the existence can either be deduced from the case $A = W$, or from the classical existence of Stiefel–Whitney maps. For $A = W$, we define P^d piecewise on quadratic forms, using (9.2) for P^d_m in each dimension m. We see immediately from the definition that $P^d_1(\langle a\rangle)$ is 1 if $d = 0$, $\langle\langle a\rangle\rangle$ if $d = 1$ and 0 if $d \geqslant 2$. The fact that this extends to a group morphism $\mathrm{GW}(K) \to \Lambda(\mathrm{GW}(K))$ can be deduced using the universal property of Grothendieck groups if we can show that for any $q \in \mathrm{Quad}_m(K)$, $q' \in \mathrm{Quad}_n(K)$, we have

$$P^d_{m+n}(q + q') = \sum_{k=0}^{d} P_m(q)P_n(q').$$

And indeed we find

$$\begin{aligned}
\sum_{k=0}^{d} P_m(q)P_n(q') &= \sum_{k=0}^{d}\sum_{i=0}^{k}\sum_{j=0}^{d-k} (-1)^{i+j} \binom{m-i}{k-i}\binom{n-j}{d-k-j} \lambda^i(q)\lambda^j(q') \\
&= \sum_{l=0}^{d} (-1)^l \sum_{i+j=l} \left(\sum_{k=i}^{d-j} \binom{m-i}{k-i}\binom{n-j}{d-k-j} \right) \lambda^i(q)\lambda^j(q') \\
&= \sum_{l=0}^{d} (-1)^l \binom{m+n-l}{d-l} \sum_{i+j=l} \lambda^i(q)\lambda^j(q') = P_{m+n}(q + q').
\end{aligned}$$

From the group property we easily see that

$$h_m^d(\langle a_1, \dots, a_m\rangle) = \sum_{i_1<\cdots<i_d} \{a_{i_1}, \dots, a_{i_d}\}, \tag{9.3}$$

so $h_m^d(q) \in I^d(K)$ if $q \in \mathrm{Quad}_m(K)$. The fact that h_m^d is an invariant is obvious given the definition with the λ-powers, or can be deduced from the uniqueness statement. Finally, the fact that the h_m^d for a basis of $\mathrm{Inv}(\mathrm{Quad}_m, A)$ is a consequence of Serre's result, directly for $A = H$, and observing for $A = W$ that the transition matrix from $(P_m^d)_{0\leqslant d\leqslant m}$ to $(\lambda^d)_{0\leqslant d\leqslant m}$ is triangular unipotent. □

Remark 9.4. Note that this does not define a pre-λ-ring structure on $\mathrm{GW}(K)$ since P^1 is not the identity (indeed, $P^1(\langle a\rangle) = \langle\langle a\rangle\rangle$).

Proposition 9.5. *Let $m = 2r \in \mathbb{N}^*$, $d \in \mathbb{N}$ and $q \in \mathrm{Quad}_m(K)$. Then*

$$f_1^d(q) = \sum_{i=0}^{d} (-1)^i \binom{r-i}{d-i} \{-1\}^{d-i} h_m^i(q),$$

$$g_1^d(q) = \sum_{i=0}^{d} (-1)^i \binom{r-i-1+\lfloor \frac{d+1}{2} \rfloor}{d-i} \{-1\}^{d-i} h_m^i(q).$$

Proof. We prove the statement concerning f_1^d; the case of g_1^d may be deduced by a lengthy but straightforward computation using Proposition 4.6, or can be directly proved by the same method.

Write α_m^d for the invariant of Quad_m defined by the right-hand side of the equation. It is clear by definition that $\alpha_m^0 = 1$ coincides with f_1^0 on Quad_m. We claim that it is enough to show that for any $d \in \mathbb{N}^*$,

$$\alpha_2^d(\langle\langle 1\rangle\rangle) = 0, \tag{9.6}$$

and for any $m = 2r \in \mathbb{N}^*$, any $q \in \mathrm{Quad}_m(K)$ and any $a \in K^*$,

$$\alpha_{m+2}^d(q + \langle\langle a\rangle\rangle) = \alpha_m^d(q) + \{a\}\alpha_m^{d-1}(q). \tag{9.7}$$

Indeed, taking $a = 1$ in (9.7) shows that $\alpha_m^d(q)$ depends only on the Witt class of $q \in \mathrm{Quad}_m(K)$, so it defines an invariant $\alpha^d \in M(1)$. Then (9.6) shows that α^d is normalized, and (9.7) shows that $(\alpha^d)^+ = \alpha^{d-1}$, so by an immediate induction, $\alpha^d = f_1^d$.

From (9.3) we easily see that $h_2^0(\langle\langle a\rangle\rangle) = 1$, $h_2^1(\langle\langle a\rangle\rangle) = \{-a\}$, and $h_2^i(\langle\langle a\rangle\rangle) = 0$ if $i \geqslant 2$. Thus $\alpha_2^d(\langle\langle 1\rangle\rangle) = \{-1\}^d - \{-1\}^{d-1} \cdot \{-1\} = 0$, which shows (9.6).

Furthermore, if $i \in \mathbb{N}$ and $q \in \mathrm{Quad}_m(K)$,

$$\begin{aligned} h_{m+2}^i(q + \langle\langle a\rangle\rangle) &= h_m^i(q) + \{-a\}h_m^{i-1}(q) \\ &= \big(h_m^i(q) + \{-1\}h_m^{i-1}(q)\big) - \{a\}h_m^{i-1}(q) \end{aligned} \tag{9.8}$$

(where by convention $h_m^{-1} = 0$). Therefore,

$$\begin{aligned}
\alpha_{m+2}(q + \langle\langle a \rangle\rangle) &= \sum_{i=0}^{d} (-1)^i \binom{r+1-i}{d-i} \{-1\}^{d-i} \big(h_m^i(q) + \{-1\} h_m^{i-1}(q)\big) \\
&\qquad - \{a\} \sum_{i=0}^{d} (-1)^i \binom{r+1-i}{d-i} \{-1\}^{d-i} h_m^{i-1}(q) \\
&= \sum_{i=0}^{d-1} \left((-1)^i \binom{r-i+1}{d-i} + (-1)^{i+1} \binom{r-i}{d-i-1} \right) \{-1\}^{d-i} h_m^i(q) \\
&\qquad + (-1)^d h_m^d(q) - \{a\} \sum_{i=0}^{d-1} (-1)^{i+1} \binom{r-i}{d-i-1} \{-1\}^{d-i-1} h_m^i(q) \\
&= \sum_{i=0}^{d} (-1)^i \binom{r-i}{d-i} \{-1\}^{d-i} h_m^i(q) \\
&\qquad + \{a\} \sum_{i=0}^{d-1} (-1)^i \binom{r-i}{d-1-i} \{-1\}^{d-1-i} h_m^i(q),
\end{aligned}$$

which gives the expected formula. □

Remark 9.9. In particular, looking carefully at the binomial coefficients in the formula and remembering that $h_m^i = 0$ if $i > m$, we retrieve the fact that g_1^d is zero if $d > m$ (recall Corollary 4.8). On the other hand, we see that f_1^d can be nonzero for arbitrarily high values of d, even for fixed m.

Remark 9.10. If -1 is a square in k, then $f_1^d = g_1^d = h_m^d$ on Quad_m for any even $m \in \mathbb{N}^*$.

Corollary 9.11. *For any even $m \in \mathbb{N}^*$, the restrictions of f_1^d (or g_1^d) for $0 \leqslant d \leqslant m$ form an $A(k)$-basis of* $\mathrm{Inv}(\mathrm{Quad}_m, A)$. *In particular, any invariant of* Quad_m *can be extended to I.*

Remark 9.12. Serre also describes the cohomological invariants of $\mathrm{Quad}_{m,\delta}$, meaning of forms with prescribed determinant δ, and in particular this gives a description of invariants of $\mathrm{Quad}_m \cap I^2$. They are given by Stiefel–Whitney classes, plus one invariant that does not extend to Quad_m in general. Since any invariant of I^2 extends to I, this shows that there are invariants of $\mathrm{Quad}_m \cap I^2$ that do not extend to I^2.

There are also examples in the literature of some invariants of $\mathrm{Quad}_m \cap I^3$ that one can show, using the results in this article, do not extend to I^3 (for instance the invariant a_5 mentioned in Section 11).

Remark 9.13. Let us consider the cohomological invariants of $\mathrm{Quad}_m /{\sim}$ (the similarity classes of quadratic forms of dimension n). This is of course the same thing as an invariant of Quad_m which is constant on similarity classes, so according to Corollary 9.11 any such invariant is a unique combination of the $v_d^{(1)}$ with $0 \leqslant d \leqslant m$.

Now Corollary 7.8 shows that such a combination is constant on similarity classes if and only if the only d that appear are odd. This is exactly the description that Rost gives in [1998, Lemma 2], where he proves that any invariant of $\mathrm{Quad}_m/\!\sim$ is a unique combination of invariants he calls v_{2i+1}, and a simple computation shows that $v_{2i+1} = v_{2i+1}^{(1)}$.

On the other hand, our present tools cannot a priori describe all cohomological invariants of similarity classes in $\mathrm{Quad}_m \cap I^2$, since not all invariants of $\mathrm{Quad}_m \cap I^2$ extend to I^2. What we can say from the previous remark and Corollary 7.8 is that those which do extend to I^2 can be uniquely written as $\sum_{d=0}^{r} a_d \cup v_{2d}^{(2)}$ with $(-1) \cup a_d = 0$ if $d > 0$ is even. However, Rost describes in [1998, Theorem 6] the invariants of similarity classes $\mathrm{Quad}_m \cap I^2$, and proves that they are combinations of invariants η_d. It turns out that $\eta_d = v_{2d}^{(2)}$, so this shows that even though some invariants of isometry classes in $\mathrm{Quad}_m \cap I^2$ do not extend to I^2, all invariants of *similarity classes* in $\mathrm{Quad}_m \cap I^2$ do extend to I^2 (and therefore to I), and Rost's description is exactly the same as ours.

10. Operations on mod 2 cohomology

In this section we are specifically interested in cohomological invariants. It was observed by Serre that one may define some sorts of divided squares on mod 2 cohomology:

$$\begin{aligned} H^n(K,\mu_2) &\to H^{2n}(K,\mu_2)/(-1)^{n-1} \cup H^{n+1}(K,\mu_2), \\ \sum_i \alpha_i &\mapsto \sum_{i<j} \alpha_i \cup \alpha_j. \end{aligned}$$

The quotient on the right-hand side is necessary for the map to be well-defined. Similarly, one may define higher divided powers:

$$\begin{aligned} H^n(K,\mu_2) &\to H^{dn}(K,\mu_2)/(-1)^{n-1} \cup H^{(d-1)n+1}(K,\mu_2), \\ \sum_i \alpha_i &\mapsto \sum_{i_1<\cdots<i_d} \alpha_{i_1} \cup \cdots \cup \alpha_{i_d}. \end{aligned}$$

On the other hand, Vial [2009] characterizes natural operations

$$H^n(K,\mu_2) \to H^*(K,\mu_2)$$

(his statement is formulated for mod 2 Milnor K-theory, which is equivalent according to the resolution of Milnor's conjecture). The precise statement, slightly reformulated, is the following (the original statement forgets to explicitly assume that operations must have uniformly bounded degree):

Proposition 10.1 [Vial 2009, Theorem 2]. *If $n \in \mathbb{N}^*$, the $H^*(k,\mu_2)$-module of operations $H^n(K,\mu_2) \to H^*(K,\mu_2)$ with uniformly bounded degree is*

$$H^*(k,\mu_2)\cdot 1 \oplus H^*(k,\mu_2)\cdot \mathrm{Id} \oplus \bigoplus_{d\in\mathbb{N}} \mathrm{Ker}(\tau_n)\cdot \theta_d,$$

where $\tau_n : H^*(k,\mu_2) \to H^*(k,\mu_2)$ *is defined by* $\tau_n(x) = (-1)^{n-1} \cup x$, *and if* $a \in \mathrm{Ker}(\tau_n)$, *then*

$$a \cdot \theta_d\Big(\sum_{1\leqslant i\leqslant r} x_i\Big) = a \cdot \sum_{i_1<\cdots<i_d} x_{i_1} \cup \cdots \cup x_{i_d},$$

where the x_i *are symbols.*

Note that the "divided power operation" θ_d is not defined on its own, but $a \cdot \theta_d$ is well-defined when $a \in \mathrm{Ker}(\tau_n)$. This is similar to how for Serre's operations it was necessary to consider some quotient on the right-hand side of the map; here one has to put some restriction on the left-hand side, in both cases to annihilate appropriate powers of the symbol $(-1) \in H^1(K,\mu_2)$. The remarkable phenomenon is that when we work on the level of I^n, we can lift those θ_d with no restriction: this is our $u^{(n)}_{nd}$.

Moreover, it is not too difficult to retrieve Vial's theorem using our results about invariants of I^n: operations on $H^n(K,\mu_2)$ are none other than invariants $\alpha \in M(n)$ (with $A = H$) such that

$$\alpha(q+\varphi) = \alpha(q) \quad \text{for all } q \in I^n(K), \varphi \in \mathrm{Pf}_{n+1}(K). \tag{10.2}$$

Consider the following easy lemma:

Lemma 10.3. *Let* $n \in \mathbb{N}^*$, *and let us restrict to* $A = H$. *For any* $\alpha \in M(n)$, *any* $q \in I^n(K)$ *and any* $\varphi \in \mathrm{Pf}_{n+1}(K)$, *we have*

$$\alpha(q+\varphi) = \alpha(q) + (-1)^{n-1} \cup e_{n+1}(\varphi) \cup \alpha^{++}(q).$$

Proof. Up to taking linear combinations, we may restrict to the case of $\alpha = u^{(n)}_{nd}$. Using Corollary 6.4, we see that $u^{(n)}_{nd}(\varphi)$ is 1 if $d = 0$, $(-1)^{n-1} \cup e_{n+1}(\varphi)$ if $d = 2$, and 0 otherwise. Then using the sum formula for $u^{(n)}_{nd}$ we find

$$u^{(n)}_{nd}(q+\varphi) = u^{(n)}_{nd}(q) + (-1)^{n-1} \cup e_{n+1}(\varphi) \cup u^{(n)}_{n(d-2)}(q). \qquad \square$$

Then $\alpha \in M(n)$ satisfies condition (10.2) if and only if $(-1)^{n-1} \cup \alpha^{++} = 0$, which precisely means that if we write $\alpha = \sum_d a_d \cup u^{(n)}_{nd}$ then, for $d \geqslant 2$, $a_d \in \mathrm{Ker}(\tau_n)$, and we indeed retrieve Vial's description.

11. Invariants of semifactorized forms

Garibaldi [2009, §20] defines a cohomological invariant on $\mathrm{Quad}_{12} \cap I^3$ in the following way: any such form can be written $q = \langle\langle c\rangle\rangle q'$, where $q' \in I^2(K)$, and we set $a_5(q) = e_5(\langle\langle c\rangle\rangle \bar{\pi}^2_2(q')) = (c) \cup u^{(2)}_4(q')$ (using our notation). Of course, the nontrivial ingredient is that $\langle\langle c\rangle\rangle \bar{\pi}^2_2(q')$ is actually independent of the decomposition of q.

This construction does not correspond to any of the tools we developed so far, since it does not give an invariant of I^3. However, it is easy to see that the construction works for any Witt class $q \in I^3(K)$ that factorizes as $q = \langle\langle c\rangle\rangle q'$. This leads us to the more general definition:

Definition 11.1. Let $n \in \mathbb{N}^*$ and $r \in \mathbb{N}$ such that $r \leqslant n$. We set

$$I^{n,r}(K) = \{\varphi \cdot q \mid \varphi \in \mathrm{Pf}_r(K),\ q \in I^{n-r}(K)\}.$$

We also define $M(n,r) = \mathrm{Inv}(I^{n,r}, A)$, and similarly $M_0(n,r)$, $M^{\geqslant d}(n,r)$ and $M_0^{\geqslant d}(n,r)$. In particular, $I^{n,0} = I^n$, so $M(n,0) = M(n)$ and so on.

Remark 11.2. A consequence of Milnor's conjecture proved in [Elman et al. 2008, Theorem 41.7] is that $I^{n,r}(K) = I^{r,r}(K) \cap I^n(K)$, so in particular

$$I^{n,r}(K) \cap I^{n+1}(K) = I^{n+1,r}(K).$$

Clearly, if $(m,s) \geqslant (n,r)$, then $I^{m,s}(K) \subset I^{n,r}(K)$, so we have a restriction morphism

$$\rho_{(n,r),(m,s)} : M(n,r) \to M(m,s), \quad \alpha \mapsto \alpha_{|I^{m,s}},$$

which is a morphism of filtered $A(k)$-algebras, and sends $M_0(n,r)$ to $M_0(m,s)$. In particular, when $r = s = 0$, we retrieve the restriction morphism $\rho_{n,m}$ defined in (6.1). We usually drop the indexes and simply write $\rho : M(n,r) \to M(m,s)$, since the indexes can be inferred from the source and target modules.

We can also define a morphism that goes in the other direction:

Proposition 11.3. *Let $n, r, t \in \mathbb{N}$ with $t \leqslant r < n$. There is a unique morphism of filtered $A(k)$-modules*

$$\Delta^t_{(n,r)} : M(n,r) \to M(n-t, r-t)[-t], \quad \alpha \mapsto \alpha^{(t)},$$

such that $\alpha^{(t)}(0) = \alpha(0)$, and if $\alpha \in M_0(n,r)$ then

$$\alpha(\varphi \cdot q) = f_t(\varphi) \cdot \alpha^{(t)}(q)$$

for any $\varphi \in \mathrm{Pf}_t(K)$ and $q \in I^{n-t,r-t}(K)$. Furthermore, $\Delta^t_{(n,r)}$ is injective.

Proof. Since $M(n,r) = A(k) \oplus M_0(n,r)$, this piecewise definition of $\Delta^t_{(n,r)}$ determines the whole function. Let $\alpha \in M_0^{\geqslant d}(n,r)$ and $q \in I^{n-t,r-t}(K)$. Then $\varphi \mapsto \alpha(\varphi \cdot q)$ defines an invariant of Pf_t over K with values in $A^{\geqslant d}$. Using Lemma 0.5, there are unique $x(q), y(q) \in A(K)$ such that

$$\alpha(\varphi \cdot q) = x(q) + f_t(\varphi) \cdot y(q)$$

and by uniqueness those are invariants of $I^{n-t,r-t}$, with $x = \alpha(0) = 0$. We then set $\alpha^{(t)} := y$. Furthermore, using Lemma 0.4, we see that $y(q) \in A^{\geqslant d-t}(K)$, so

$\alpha^{(t)} \in M_0^{\geqslant d-t}(n-t, r-t)$. The injectivity is clear since any element of $I^{n,r}(K)$ is of the form φq with φ and q as in the statement, and $\alpha(\varphi q)$ is determined by $\alpha^{(t)}$. □

We usually drop the indexes and simply write $\Delta^t : M(n, r) \to M(n-t, r-t)[-t]$. Using this notation, it is clear by definition that $\Delta^t \circ \Delta^{t'} = \Delta^{t+t'}$. The natural question is then as follows:

Question. What is the image of $\Delta^t : M(n+t, r+t) \to M(n, r)$?

This can be rephrased to ask for which $\beta \in M_0(n, r)$ is it true that for all $\varphi \in \mathrm{Pf}_t(K)$ and $q \in I^{n,r}(K)$, $f_t(\varphi)\beta(q)$ only depends on φq? With this point of view, the existence of the invariant a_5 given at the beginning of the section (which is [Garibaldi 2009, Corollary 20.7]) is exactly equivalent to the fact that $f_2^2 \in M(2, 0)$ is in the image of $\Delta^1 : M(3, 1) \to M(2, 0)$. The main result of the section is a generalization of this fact:

Theorem 11.4. *For any $n \in \mathbb{N}^*$, $\Delta^1 : M_0(n+1, 1) \to M_0(n)[-1]$ is an isomorphism of filtered $A(k)$-modules.*

Remark 11.5. This means that $\Delta^1 : M(n+1, 1) \to M(n)[-1]$ is a module isomorphism, but it is not a *filtered* module isomorphism, since it is the identity on the constant components, and while the identity is a bijective filtered morphism from $A(k)$ to $A(k)[-1]$, it is of course not a filtered isomorphism.

Before we prove Theorem 11.4, we construct a common generalization of ρ and Δ^t, which allows us to make simple statements about the general properties of both those morphisms. Most of that is not useful for the proof of the theorem, but has some independent interest.

Definition 11.6. Let $m, n \in \mathbb{N}^*$ and $r, s \in \mathbb{N}$ be such that $r < n$ and $s < m$. We say that a filtered $A(k)$-module morphism $M(n, r) \to M(m, s)[-t]$ is of type Ω^t if it is a composition of morphisms $\omega_i : M(n_i, r_i)[-a_i] \to M(n_{i+1}, r_{i+1})[-a_i - t_i]$ for $i = 0, \ldots, d$, with $(n_0, r_0) = (n, r)$, $a_0 = 0$, $(n_{d+1}, r_{d+1}) = (m, s)$, $t = \sum_i t_i$, and ω_i is either ρ (so $t_i = 0$) or Δ^{t_i}.

In particular, we define ω of type Ω^1:

$$\omega : M(n, r) \xrightarrow{\rho} M(n+1, r+1) \xrightarrow{\Delta^1} M(n, r)[-1].$$

Remark 11.7. It is not hard to see that there is a morphism $M(n, r) \to M(m, s)[-t]$ of type Ω^t if and only if $t \geqslant n - m$ and $t \geqslant r - s$.

Proposition 11.8. *Let $m, n \in \mathbb{N}^*$ and $r, s \in \mathbb{N}$ be such that $r < n$ and $s < m$, and let $t \in \mathbb{N}$ be such that $t \geqslant n - m$ and $t \geqslant r - s$. Then there is exactly one morphism $M(n, r) \to M(m, s)[-t]$ of type Ω^t, and we call it simply Ω^t. The morphism $\Omega^t : M(n, r) \to M(n, r)[-t]$ is ω^t.*

In particular, let $t' \geqslant t$. Then the following diagram of filtered $A(k)$-modules commutes:

$$\begin{tikzcd} & M(n,r) \arrow[dl, "\Omega^t"'] \arrow[dr, "\Omega^{t'}"] & \\ M(m,s)[-t] \arrow[rr, "\omega^{t'-t}"] & & M(m,s)[-t'] \end{tikzcd}$$

Proof. The only thing to prove is that there is at most one morphism of type Ω^t. The fact that $\Omega^t = \omega^t$ then follows, since ω^t is of type Ω^t by definition, and the commutativity of the diagram comes from the fact that both compositions are of type $\Omega^{t'}$.

To show this uniqueness, it is enough to show that the following diagram commutes whenever it makes sense:

$$\begin{tikzcd} M(n,r) \arrow[r, "\rho"] \arrow[d, "\Delta^t"'] & M(m,s) \arrow[d, "\Delta^t"] \\ M(n-t, r-t)[-t] \arrow[r, "\rho"] & M(m-t, s-t)[-t] \end{tikzcd}$$

Indeed, if we can prove this, then we can show by induction on the length of the composition that in the definition of a morphism of type Ω^t we can always assume that the first morphisms are all of the form ρ, and the remaining ones are all of the form Δ^{t_i}. But then the result is clear, since a composition of restriction morphisms is a restriction morphism, and $\Delta^t \circ \Delta^s = \Delta^{t+s}$ (with the only indices that make sense), so the morphism is entirely characterized by its source, its target and t.

We now show that the diagram commutes. Let $\alpha \in M(n,r)$. Since all morphisms are the identity on the constant components, we may assume $\alpha \in M_0(n,r)$. Let us write $\beta = (\alpha^{(t)})_{|I^{m-t,s-t}}$, and take $\varphi \in \mathrm{Pf}_t(K)$, $\psi \in \mathrm{Pf}_{s-t}(K)$ and $q \in I^{m-s}(K)$. We can set $\psi = \psi_1\psi_2$ with $\psi_1 \in \mathrm{Pf}_{r-t}(K)$ and $\psi_2 \in \mathrm{Pf}_{s-r}(K)$; then if $q' = \psi_2 q \in I^{m-r}(K)$, we have

$$\alpha(\varphi\psi q) = \alpha(\varphi\psi_1 q') = f_t(\varphi)\alpha^{(t)}(\psi_1 q') = f_t(\varphi)\beta(\psi q),$$

which shows that $\beta = (\alpha_{|I^{m,s}})^{(t)}$. □

Example 11.9. The morphism $\Omega^0 : M(n,r) \to M(m,s)$ exists when $(m,s) \geqslant (n,r)$, and it is the restriction morphism ρ. The morphism $\Omega^t : M(n,r) \to M(n-t, r-t)[-t]$ exists when $t \leqslant r$, and it is Δ^t.

Example 11.10. There is a morphism $\Omega^t : M(n) \to M(m)[-t]$ when $t \geqslant n-m$, and if $n = m$ it is ω^t, with $\omega : M(n) \to M(n)[-1]$.

We can now collect some basic properties of the morphisms Ω^t.

Proposition 11.11. *Let $n, m, r, s, t \in \mathbb{N}$ be as in Proposition 11.8. Then for any $\alpha, \beta \in M_0(n,r)$, we have*

$$\Omega^t(\alpha\beta) = \{-1\}^t \Omega^t(\alpha)\Omega^t(\beta).$$

Proof. Since the restriction morphisms obviously preserve the product of invariants, we may assume that $\Omega^t = \Delta^t$. Then for any $\varphi \in \mathrm{Pf}_t(K)$, $\psi \in \mathrm{Pf}_{r-t}(K)$ and $q \in I^{n-r}(K)$, we have

$$(\alpha\beta)(\varphi\psi q) = (f_t(\varphi)\alpha^{(t)}(\psi q))(f_t(\varphi)\beta^{(t)}(\psi q)) = \{-1\}^t f_t(\varphi)(\alpha^{(t)}\beta^{(t)})(\psi q),$$

hence the result. □

We may note from Proposition 7.1 that we have well-defined filtered morphisms

$$\Psi : M(n,r) \to M(n,r)[-1]$$

for any $n, r \in \mathbb{N}$ such that $r < n$.

Proposition 11.12. *Let $n, m, r, s, t \in \mathbb{N}$ be as in Proposition 11.8. Then the following diagram of filtered $A(k)$-modules commutes*:

$$\begin{tikzcd} M(n,r) \arrow[r, "\Omega^t"] \arrow[d, "\Psi"'] & M(m,s)[-t] \arrow[d, "\Psi"] \\ M(n,r)[-1] \arrow[r, "\Omega^t"] & M(m,s)[-t-1] \end{tikzcd}$$

Proof. The definition of Ψ makes it clear that it commutes with restriction morphisms, since it is defined on the whole $\mathrm{Inv}(W, A)$. Thus we may assume $\Omega^t = \Delta^t$. Let $\alpha \in M(n,r)$, $\varphi \in \mathrm{Pf}_t(K)$, $\psi \in \mathrm{Pf}_{r-t}(K)$, $q \in I^{n-r}(K)$ and $\lambda \in K^*$. Then

$$\alpha(\langle\lambda\rangle\varphi\psi q) = f_t(\varphi)\alpha^{(t)}(\langle\lambda\rangle\psi q) = f_t(\varphi)\alpha^{(t)}(\psi q) + f_t(\varphi)\{\lambda\}\widetilde{\alpha^{(t)}}(\psi q),$$

but also

$$\alpha(\langle\lambda\rangle\varphi\psi q) = \alpha(\varphi\psi q) + \{\lambda\}\widetilde{\alpha}(\varphi\psi q) = f_t(\varphi)\alpha^{(t)}(\psi q) + \{\lambda\} f_t(\varphi)\widetilde{\alpha}^{(t)}(\psi q),$$

which gives $\widetilde{\alpha^{(t)}} = \widetilde{\alpha}^{(t)}$. □

Since we saw in Corollary 6.3 that Φ^+ is far from commuting with the restriction morphisms, we cannot expect such a good compatibility with the morphisms Ω^t, but we still get the following:

Proposition 11.13. *Let $n \in \mathbb{N}^*$ and let $t \in \mathbb{N}$ be such that $t < n$. Then the following diagram of filtered $A(k)$-modules commutes for any $\varepsilon = \pm 1$*:

$$\begin{tikzcd} M(n) \arrow[r, "\Omega^t"] \arrow[d, "\Phi^\varepsilon"'] & M(n-t)[-t] \arrow[d, "\Phi^\varepsilon"] \\ M(n)[-n] \arrow[r, "\{-1\}^t\Omega^t"] & M(n-t)[-n] \end{tikzcd}$$

Proof. The diagram obviously commutes for the constant components (since we find 0 in both cases), so we may consider $\alpha \in M_0(n)$. Let $\varphi \in \mathrm{Pf}_t(K)$, $\psi \in \mathrm{Pf}_{n-t}(K)$ and $q \in I^{n-t}(K)$. Then

$$\begin{aligned}\alpha(\varphi(q+\varepsilon\psi)) &= \alpha(\varphi q) + \varepsilon f_n(\varphi\psi)\alpha^{\varepsilon}(\varphi q)\\ &= f_t(\varphi)\alpha^{(t)}(q) + \varepsilon\{-1\}^t f_n(\varphi\psi)(\alpha^+)^{(t)}(q)\end{aligned}$$

as well as

$$\begin{aligned}\alpha(\varphi(q+\varepsilon\psi)) &= f_t(\varphi)\alpha^{(t)}(q+\varepsilon\psi)\\ &= f_r(\varphi)\alpha^{(r)}(q) + \varepsilon f_r(\varphi) f_{n-r}(\psi)(\alpha^{(r)})^+(q),\end{aligned}$$

which proves that $((\alpha_{|I^{n,t}})^{(t)})^+ = \{-1\}^t(\alpha^+)^{(t)}_{|I^{n,t}}$. □

Corollary 11.14. *Let $n, t \in \mathbb{N}$ be such that $t<n$. Then for any $d \in \mathbb{N}^*$, the morphism $\Omega^t : M(n) \to M(n-t)[-t]$ satisfies*

$$\Omega^t(f_n^d) = \{-1\}^{t(d-1)} f_{n-t}^d.$$

In particular, if $\varphi \in \mathrm{Pf}_t(K)$ and $q \in I^n(K)$ is a multiple of φ, then $f_n^d(q)$ is a multiple of $f_t(\varphi)$.

Proof. The formula follows from an induction on d, using Proposition 11.13. For the last statement, note that according to Remark 11.2, there is $q' \in I^{n-t}(K)$ such that $q = \varphi q'$. Then according to the formula,

$$f_n^d(q) = f_n^d(\varphi q') = \{-1\}^{t(d-1)} f_t(\varphi) f_{n-t}^d(q'). \qquad □$$

We now turn to the proof of Theorem 11.4. We first need a preliminary lemma.

Lemma 11.15. *Let $a, b \in K^*$, and consider $q \in \hat{I}(K)$ of the form*

$$q = \sum_{i=1}^{r} \langle x_i\rangle\langle|c_i|\rangle,$$

where c_i is represented by $\langle\langle ab\rangle\rangle$. Then for any $k \in \mathbb{N}^*$,

$$\langle\langle a\rangle\rangle\lambda^k(q) = \langle\langle b\rangle\rangle\lambda^k(q).$$

In particular, for any $n, d \in \mathbb{N}^*$, $\langle\langle a\rangle\rangle\pi_n^d(q) = \langle\langle b\rangle\rangle\pi_n^d(q)$.

Proof. We have

$$\lambda^k(q) = \sum_{d_1+\cdots+d_r=k} \lambda^{d_1}(\langle x_1\rangle\langle|c_1|\rangle)\cdots\lambda^{d_r}(\langle x_r\rangle\langle|c_r|\rangle).$$

Now at least one of the d_i is nonzero, so we may conclude since

$$\langle\langle a\rangle\rangle\lambda^{d_i}(\langle x_i\rangle\langle|c_i|\rangle) = \langle x_i^d\rangle\langle\langle a\rangle\rangle\langle|c_i|\rangle = \langle x_i^d\rangle\langle\langle b\rangle\rangle\langle|c_i|\rangle = \langle\langle b\rangle\rangle\lambda^{d_i}(\langle x_i\rangle\langle|c_i|\rangle),$$

using Lemma 1.4 and the fact that if c is represented by $\langle\langle ab\rangle\rangle$ then $\langle\langle a, c\rangle\rangle = \langle\langle b, c\rangle\rangle$. The statement about π_n^d follows since by definition π_n^d is a combination of the λ^k with $1 \leqslant k \leqslant d$. $\square$

We can finally prove the main result of the section:

Proof of Theorem 11.4. It suffices to show that f_n^d is in the image of Δ^1 for all $d \geqslant 1$, which amounts to saying that $\langle\langle a\rangle\rangle q \mapsto \{a\} f_n^d(q)$ is well-defined, in other words that if $q, q' \in I^n(K)$ and $a, b \in K^*$, then $\langle\langle a\rangle\rangle q = \langle\langle b\rangle\rangle q'$ implies $\{a\} f_n^d(q) = \{b\} f_n^d(q')$.

Assume first that $a = b$. Then according to [Elman et al. 2008, Corollary 6.23],

$$q - q' = \sum_{i \in J} \langle\langle c_i\rangle\rangle q_i,$$

where $q_i \in W(K)$ and c_i is represented by $\langle\langle a\rangle\rangle$. We may then reason by induction on $|J|$, and we are reduced to the case where $q' = q + \langle\langle c\rangle\rangle q_0$, with c represented by $\langle\langle a\rangle\rangle$. But according to Corollary 11.14, for any $k \in \mathbb{N}^*$, $f_n^k(\langle\langle c\rangle\rangle q_0)$ is divisible by $\{c\}$, so $\{a\} f_n^k(\langle\langle c\rangle\rangle q_0) = 0$. From there,

$$\{a\} f_n^d(q') = \{a\} \sum_{k=0}^{d} f_n^k(q) f_n^{d-k}(\langle\langle c\rangle\rangle q_0) = \{a\} f_n^d(q).$$

Suppose now that $a \neq b$. Then Hoffmann shows in [Garibaldi 2009, Corollary B.5] that we have

$$\langle\langle a\rangle\rangle q = \langle\langle a\rangle\rangle q_0 = \langle\langle b\rangle\rangle q_0 = \langle\langle b\rangle\rangle q',$$

where $q_0 = \sum_{i \in J} \langle x_i\rangle \langle\langle c_i\rangle\rangle \in I^n(K)$, and c_i is represented by $\langle\langle ab\rangle\rangle$. The previous discussion shows that $\{a\} f_n^d(q) = \{a\} f_n^d(q_0)$ and $\{b\} f_n^d(q) = \{b\} f_n^d(q_0)$, so it just remains to show that $\{a\} f_n^d(q_0) = \{b\} f_n^d(q_0)$ for any q_0 admitting a decomposition as above. This is a direct consequence of Lemma 11.15. $\square$

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