Lambda-operations for hermitian forms over algebras with involution of the first kind

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Introduction

In his seminal work [?], Serre defines a general notion of *invariants* of a certain class of objects over a base field: if K is a field, and H and A are two functors defined on the category of field extensions of K to the category of sets, then the set $\mathrm{Inv}(H, A)$ of invariants of H with values in A is defined as the set of natural transformations from H to A. Often, A is actually a functor to the category of rings, and $\mathrm{Inv}(H, A)$ is then naturally an $A(K)$-algebra. The idea is very simple and broad: for each extention L/K, to each element of $H(L)$ we associate an element of $A(L)$, with the only constraint that this is compatible with field extensions.

The primary source of examples is given by $H(L) = H^1(L, G(L))$ (non-abelian cohomology) where G is an algebraic group over K, and $A(L) = H^*(L, \mathbb{Z}/2\mathbb{Z})$ (mod 2 Galois cohomology) or $A(L) = W(L)$ (the Witt group of quadratic forms). This is the study of so-called (mod 2) cohomological invariants and Witt invariants of algebraic groups, which is a very active field of study. When G is the orthogonal group of some non-degenerate quadratic form of rank n, the corresponding $H(L)$ is simply the set $\mathrm{Quad}_n(L)$ of isometry classes of non-degenerate quadratic forms of rank n, thus we are looking at invariants of quadratic forms of fixed dimension.

In this special case, Serre gives a complete description of the invariants: the cohomological invariants of Quad_n are a free $H^*(K, \mathbb{Z}/2\mathbb{Z})$-module with basis the Stiefel-Whitney invariants $w_0, \dots, w_n$, and the Witt invariants are a free $W(K)$-module over the so-called λ-powers $\lambda^0, \dots, \lambda^n$. Those operations, introduced by Bourbaki in [?], can be described either explicitly given a diagonalization:

$$\lambda^d(\langle a_1, \dots, a_n\rangle) = \sum_I \langle a_I\rangle$$

where I runs over the subsets of $\{1, \dots, n\}$ of cardinal d and $a_I = \prod_{i\in I} a_i$; or it can be described more intrinsically given a bilinear space (V, b):

$$\begin{array}{rrcl} \lambda^d(b): & \Lambda^d(V)\times\Lambda^d(V) & \longrightarrow & K \\ & (u_1\wedge\cdots\wedge u_d, v_1\wedge\cdots v_d) & \longmapsto & \det\left(b(u_i, v_j)\right). \end{array}$$

Those operations, though very natural, make suprisingly few appearances in the quadratic form literature; for instance, they do not even get a passing mention in references such as [], [], [] or []. This might be in part due to the fact that they are not well-defined on the Witt ring, which is traditionally the preferred structure for working with quadratic forms, but rather on the

Grothendieck-Witt ring $GW(K)$. The appearance of the Grothendieck-Witt ring in this context should not come as a surprise: the theory of λ-rings was initiated by Grothendieck to be applied to K-theory, where it had most of its success, and $GW(K)$ is nothing but the 0th hermitian K-theory ring of K. This being said, although the fact that the K_0 of a variety is a λ-ring was shown very early on, it has only been proved for $GW(K)$ somewhat recently [].

It seems that, in the quest for cohomological invariants, the proof of the Milnor conjecture by Voevodsky has spurred some resurgence of interest for λ-operations of quadratic forms through the following strategy: since we now have access to canonical morphisms $e_n : I^n(K) \to H^n(K, \mathbb{Z}/2\mathbb{Z})$ (where $I^n(K)$ is the nth power of the fundamental ideal $I(K)$ of the Witt ring), we can define cohomological invariants by constructing invariants with values in I^n. We will say that a mod 2 cohomological invariant is *liftable* if it is obtainable this way. As a basic example, it is not difficult to show, using λ-operations, that the Stiefel-Whitney invariants are liftable. In [], we strongly rely on λ-operations to describe all cohomological invariants of I^n, which turn out to be all liftable. This was inspired by previous constructions by Rost [] (improved by Garibaldi in []), already using λ-operations to define some liftable invariants of spin groups.

When the algebraic group G is a classical group, we are led to consider invariants of central simple algebras with involution. To implement our strategy, we need to be able to associate quadratic forms to those objects. The most common such construction is given by trace forms: if (A, σ) is an algebra with involution of the first kind, we can define the trace form $T_A : x \mapsto \mathrm{Trd}_A(x^2)$, the involution trace form $T_\sigma : x \mapsto \mathrm{Trd}_A(x\sigma(x))$, its restriction T_σ^+ to the subspace of σ-symmetric elements, and its restriction T_σ^- to the subspace of anti-symmetric elements. These forms are related by $T_A = T_\sigma^+ - T_\sigma^-$ and $T_\sigma = T_\sigma^+ + T_\sigma^-$, so it is enough to know T_σ^+ and T_σ^-. They have indeed been used to define or compute some cohomological invariants, for instance in [] or [].

If (A, σ) is a central simple algebra with involution of the first kind over K, and h is an ε-hermitian form over (A, σ) (with $\varepsilon = \pm 1$), we define in this article the λ-power $\lambda^d(h)$ for any $d \in \mathbb{N}$, which is a quadratic form over K if d is even, and an ε-hermitian form over (A, σ) if d is odd. In fact, we defined in [] a graded commutative ring

$$\widetilde{GW}(A, \sigma) = GW(K) \oplus GW^{-1}(K) \oplus GW(A, \sigma) \oplus GW^{-1}(A, \sigma),$$

and we define here a graded pre-λ-ring structure on this ring.

This gives a whole new means of associating a quadratic form to an ε-hermtian form, using $h \mapsto \sum_d a_d \lambda^{2d}(h)$ for some $a_d \in W(K)$, and also to an algebra with involution (A, σ), applying this method to the canonical diagonal form $h = \langle 1 \rangle_\sigma$. We recover as a special case the trace forms mentioned above, using $\lambda^2(\langle 1 \rangle_\sigma)$ (see **??**).

Preliminaries and conventions

We fix a base field K of characteristic not 2, and we identify symmetric bilinear forms and quadratic forms over K, through $b \mapsto q_b$ with $q_b(x) = b(x, x)$. Diagonal quadratic forms are denoted $\langle a_1, \dots, a_n \rangle$, with $a_i \in K^*$.

When we say that (A, σ) is an algebra with involution over K, we mean that A is a central simple algebra over K, and that σ is an involution of the first kind on A, so σ is an anti-automorphism of K-algebra of A, with $\sigma^2 = \mathrm{Id}_A$. In general, "involution" will be synonym with "involution of the first kind". We set $\mathrm{Sym}(A,\sigma)$ for the set of symmetric elements of A, which satisfy $\sigma(a) = a$, and $\mathrm{Skew}(A,\sigma)$ for the set of anti-symmetric elements, for which $\sigma(a) = -a$. The involution σ is orthogonal if $\dim_K(\mathrm{Sym}(A,\sigma)) = n(n+1)/2$, and it is symplectic if $\dim_K(\mathrm{Sym}(A,\sigma)) = n(n-1)/2$. In particular, (K,Id) is an algebra with orthogonal involution. A quaternion algebra admits a unique symplectic involution, called its canonical involution, and we denote it by γ.

If A is a central simple algebra over K, we write $\mathrm{Trd}_A : A \to K$ for the reduced trace of A, and $\mathrm{Nrd}_A : A \to K$ for its reduced norm. They can be defined as descents of the usual trace and determinant maps on endormorphism algebras of vector spaces.

If L/K is any field extension, and X is an object (algebra, module, involution, hermitian form, etc.) over K, then $X_L = X \otimes_K L$ is the corresponding object over L, obtained by base change.

1 The mixed Grothendieck-Witt ring

In this section, we review the definitions and results from [] which are necessary for our purposes (we refer [] for the proofs and details).

Definition 1.1. *Let (A,σ) and (B,τ) be algebras with involution over K, and let $\varepsilon = \pm 1$. An ε-hermitian Morita equivalence from (B,τ) to (A,σ) is a B-A-bimodule V endowed with a regular ε-hermitian form $h : V \times V \to A$ over (A,σ) (with $\varepsilon = \pm 1$), such that the action of B on V induces a K-algebra isomorphism $B \simeq \mathrm{End}_A(V)$, under which τ is sent to the adjoint involution σ_h.*

There exists such an equivalence if and only if A and B are Brauer-equivalent; in this case, the isomorphism class of the bimodule V is unique, and if we fix such a V, the ε-hermitian form h is unique up to a multiplicative scalar: if h' is another choice, there is some $\lambda \in K^\times$ such that $h' = \langle\lambda\rangle h$. Note that $\varepsilon = 1$ when σ and τ have the same type, and $\varepsilon = -1$ when they have opposite type.

Definition 1.2. *The hermitian Brauer 2-group $\mathbf{Br}_h(K)$ of K is the category whose objects are algebras with involutions over K, and morphisms $(B,\tau) \to (A,\sigma)$ are isomorphism classes of ε-hermitian Morita equivalences from (B,τ) to (A,σ).*

The composition of $(U,g) : (C,\theta) \to (B,\tau)$ and $(V,h) : (B,\tau) \to (A,\sigma)$ is defined as $(U \otimes_B V, f)$ with

$$f(u \otimes v, u' \otimes v') = h(v, g(u,u')v').$$

Note that the identity of (A,σ) in $\mathbf{Br}_h(K)$ is the diagonal form $(A, \langle 1\rangle_\sigma)$, and that all morphisms are invertible.

Definition 1.3. *Let (A,σ) be an algebra with involution over K. The mixed Grothendieck-Witt group of (A,σ) is*

$$\widetilde{GW}(A,\sigma) = GW(K) \oplus GW^{-1}(K) \oplus GW(A,\sigma) \oplus GW^{-1}(A,\sigma).$$

It is a basic consequence of hermitian Morita theory that $\widetilde{GW}$ is a functor from $\mathbf{Br}_h(K)$ to the category of graded groups.

Theorem 1.4. *Let (A,σ) be an algebra with involution over K, and let $d,r\in\mathbb{N}$ be integers of the same parity. Then there is a canonical hermitian Morita equivalence*

$$\varphi^{d,r}_{(A,\sigma)}:(A^{\otimes d},\sigma^{\otimes d})\to(A^{\otimes r},\sigma^{\otimes r})$$

such that:

- $\varphi^{d,d}_{(A,sigma)}$ *is the identity of* $(A^{\otimes d},\sigma^{\otimes d})$*;*
- $\varphi^{r_1+r_2,s}_{(A,\sigma)}\circ(\varphi^{d_1,r_1}_{(A,\sigma)}\otimes\varphi^{d_1,r_1}_{(A,\sigma)})=\varphi^{d_1+d_2,s}_{(A,\sigma)}$*;*
- *if* $f:(B,\tau)\to(A,\sigma)$ *is any morphism in* $\mathbf{Br}_h(K)$*, then* $\varphi^{d,r}_{(A,\sigma)}\circ f^{\otimes d}=f^{\otimes r}\circ\varphi^{d,r}_{(B,\tau)}$.

The morphism $\varphi^{2,0}_{(A,\sigma)}$ is given by the bilinear space (A,T_σ), where we recall that $T_\sigma(x,y)=\mathrm{Trd}_A(\sigma(x),y)$.

Definition 1.5. *Let (A,σ) be an algebra with involution over K, let $\varepsilon,\varepsilon'\in\{1,-1\}$, and let $i,j\in\{0,1\}$. Then we define a product*

$$GW^{\varepsilon}(A^{\otimes i},\sigma^{\otimes i})\times GW^{\varepsilon'}(A^{\otimes j},\sigma^{\otimes j})\to GW^{\varepsilon\varepsilon'}(A^{\otimes r},\sigma^{\otimes r})$$

where $r\in\{0,1\}$ is the remainder of $i+j$ mod 2, using the isomorphism

$$GW^{\varepsilon\varepsilon'}(A^{\otimes i+j},\sigma^{\otimes i+j})\xrightarrow{\sim}GW^{\varepsilon\varepsilon'}(A^{\otimes r},\sigma^{\otimes r})$$

induced by $\varphi^{i+j,r}_{(A,\sigma)}$.

This extends obviously to an internal product on $\widetilde{GW}(A,\sigma)$.

Theorem 1.6. *The product defined above makes $\widetilde{GW}$ into a functor from $\mathbf{Br}_h(K)$ to the category of graded commutative rings.*

2 Alternating powers of a module

If V is K-vector space, since $char(K)\neq 2$ we may see the exterior power $\Lambda^d(V)$ in two different ways: either as a quotient of $V^{\otimes d}$ (which is the canonical construction), or as a subspace. Precisely, for any $\pi\in\mathfrak{S}_d$ we set

$$\begin{array}{rccc} g_\pi: & V^{\otimes d} & \longrightarrow & V^{\otimes d} \\ & v_1\otimes\cdots\otimes v_d & \longmapsto & v_{\pi^{-1}(1)}\otimes\cdots\otimes v_{\pi^{-1}(d)}, \end{array}$$

and we define the anti-symmetrization map

$$s_d=\sum_{\pi\in\mathfrak{S}_d}(-1)^\pi g_\pi$$

where $(-1)^\pi$ is the signature of the permutation π. Then the map s_d is alternating, so by universal property we get an induced map $\Lambda^d(V)\to\mathrm{Alt}^d(V)$, where $\mathrm{Alt}^d(V)\subset V^{\otimes d}$ is the image of s_d, and a classical result of linear algebra states that this is an isomorphism, that may be explicited as $v_1\wedge\cdots\wedge v_d\mapsto s_d(v_1\otimes\cdots\otimes v_d)$.

Since the maps g_π are precisely the elements $g_A(\pi)$ in the case $A = \mathrm{End}_K(V)$ (see remark **??**), we may generalize this construction in the non-split case. Thus we define, as in [**?**, §10.A], the anti-symmetrisation element $s_{d,A} \in A^{\otimes d}$ by:

$$s_{d,A} = \sum_{\pi \in \mathfrak{S}_d} (-1)^\pi g_A(\pi). \tag{1}$$

Note that $s_{0,A} = 1 \in K$, $s_{1,A} = 1 \in A$, and $s_{2,A} = 1 - g_A \in A \otimes_K A$.

Definition 2.1. *Let A be a central simple algebra over K, let V be a right A-module, with $B = \mathrm{End}_A(V)$, and let $d \in \mathbb{N}$. We set*

$$\mathrm{Alt}^d(V) = s_{d,B} V^{\otimes d} \subset V^{\otimes d}$$

as a right $A^{\otimes d}$-module, with in particular $\mathrm{Alt}^0(V) = K$ and $\mathrm{Alt}^1(V) = V$.

By construction, we retrieve the case of vector spaces discussed above, when $A = K$. In general, we have:

Proposition 2.2. *Let A be a central simple algebra over K, let V be a right A-module, and let $d \in \mathbb{N}$. Then*

$$\mathrm{rdim}_{A^{\otimes d}}(\mathrm{Alt}^d(V)) = \binom{\mathrm{rdim}_A(V)}{d}.$$

Proof. It is enough to check this when A is split, in which case $A = \mathrm{End}_K(U)$ for some K-vector space U, and $V \simeq W \otimes_K U$ for some K-vector space W, so that $\mathrm{End}_A(V) \simeq \mathrm{End}_K(W)$. Then by construction $\mathrm{Alt}^d(V) \simeq \Lambda^d(W) \otimes_K U^{\otimes d}$, so $\mathrm{End}_{A^{\otimes d}}(V) \simeq \mathrm{End}_K(\lambda^d(W))$. Thus if $n = \dim_K(W)$, then $\mathrm{rdim}_A(V) = n$ and $\mathrm{rdim}_{A^{\otimes d}}(\mathrm{Alt}^d(V)) = \binom{n}{d}$. □

In particular, if $d > \mathrm{rdim}_A(V)$ then $\mathrm{Alt}^d(V) = \{0\}$. This means that we have to be a little careful if we want to allow arbitrary $d \in \mathbb{N}$, since most of our results have been stated for non-zero modules. Note that this means that if $d > \deg(A)$ then $s_{d,A} = 0$.

3 The shuffle product

We start by recalling some elementary results about symmetric groups and shuffles. Let $d \in \mathbb{N}$, and $p, q \in \mathbb{N}$ such that $p + q = d$. Then we can define the Young subgroup $\mathfrak{S}_{p,q} \subset \mathfrak{S}_d$, which contains the permutations that preserve the sets $\{1, \dots, p\}$ and $\{p+1, \dots, p+q\}$. There is a natural isomorphism $\mathfrak{S}_{p,q} \simeq \mathfrak{S}_p \times \mathfrak{S}_q$, such that the restriction of the signature of $\mathfrak{S}_d$ corresponds to the product of the signatures on $\mathfrak{S}_p$ and $\mathfrak{S}_q$.

Lemma 3.1. *Let V_1 and V_2 be right A-modules, and let $B_i = \mathrm{End}_A(V_i)$ and $B = \mathrm{End}_A(V_1 \oplus V_2)$. Take $\pi \in \mathfrak{S}_{p,q}$, corresponding to the product of $\pi_1 \in \mathfrak{S}_p$ and $\pi_2 \in \mathfrak{S}_q$. Let $x \in V_1^{\otimes p}$ and $y \in V_2^{\otimes q}$. Then:*

$$g_B(\pi) \cdot (x \otimes y) = (g_{B_1}(\pi_1) \cdot x) \otimes (g_{B_2}(\pi_2) \cdot y).$$

Proof. It is enough to treat the case where A is split, in which case it is clear by construction. □

We may also define the set of (p,q)-shuffles $Sh(p,q) \subset \mathfrak{S}_d$, which are the permutations that are increasing functions when restricted to $\{1,\dots,p\}$ and $\{p+1,\dots,p+q\}$.

Lemma 3.2. *Any element of $\mathfrak{S}_d$ can be written in a unique way as $\pi\sigma$, with $\pi \in Sh(p,q)$ and $\sigma \in \mathfrak{S}_{p,q}$.*

Proof. Let $\tau \in \mathfrak{S}_d$. We set $\sigma_1 \in \mathfrak{S}_p$ and $\sigma_2 \in \mathfrak{S}_q$ defined by $\tau(\sigma_1^{-1}(1)) < \cdots < \tau(\sigma_1^{-1}(p))$ and $\tau(\sigma_2^{-1}(p+1)) < \cdots < \tau(\sigma_2^{-1}(p+q))$; in other words, σ_1 is obtained by ordering $\tau(1),\dots,\tau(p)$ in increasing order, and likewise for σ_2 with $\tau(p+1),\dots,\tau(p+q)$. We take $\sigma \in \mathfrak{S}_{p,q}$ corresponding to (σ_1,σ_2), and $\pi = \tau\sigma^{-1}$. Then by construction $\pi(1) < \cdots < \pi(p)$ and $\pi(p+1) < \cdots < \pi(p+q)$, so $\pi \in Sh(p,q)$.

If we have another decomposition $\tau = \sigma'\pi'$, then since $\pi' \in Sh(p,q)$ we must have $\tau((\sigma_1')^{-1}(1)) < \cdots < \tau((\sigma_1')^{-1}(p))$ and $\tau((\sigma_2')^{-1}(p+1)) < \cdots < \tau((\sigma')_2^{-1}(p+q))$, so $\sigma' = \sigma$ (and $\pi' = \pi$). □

As a consequence, we get:

Lemma 3.3. *Let A be a central simple algebra over K. Then the element*

$$sh_{p,q,A} = \sum_{\pi \in Sh(p,q)} (-1)^\pi g_A(\pi) \in A^{\otimes d}$$

satisfies

$$sh_{p,q,A} \cdot (s_{p,A} \otimes s_{q,A}) = s_{d,A}.$$

Proof. We have, using lemmas 3.1 and 3.2:

$$\begin{aligned} sh_{p,q}(s_p \otimes s_q) &= \sum_{\pi \in Sh(p,q)} (-1)^\pi g(\pi)(s_p \otimes s_q) \\ &= \sum_{\pi \in Sh(p,q)} (-1)^\pi g(\pi) \left(\sum_{\sigma \in \mathfrak{S}_{p,q}} (-1)^\sigma g(\sigma) \right) \\ &= \sum_{\pi \in Sh(p,q), \sigma \in \mathfrak{S}_{p,q}} (-1)^{\pi\sigma} g(\pi\sigma) \\ &= s_d. \end{aligned}$$ □

Let A be a central simple algebra over K, and V a A-module, with $B = \mathrm{End}_V(A)$. If $p+q = d$, we define a $A^{\otimes d}$-module morphism

$$V^{\otimes p} \otimes_K V^{\otimes q} \to V^{\otimes d},$$

called the *shuffle product* and denoted $x\#y$, by

$$x\#y = sh_{p,q,B} \cdot (x \otimes y) \tag{2}$$

where $sh_{p,q,B}$ is defined in lemma 3.3.

We easily see be definition that the shuffle product is associative and alternating, and in particular anti-symmetric.

Proposition 3.4. *The shuffle product induces a commutative diagram*

$$\begin{array}{ccc} V^{\otimes p} \otimes_K V^{\otimes q} & \xrightarrow{\otimes} & V^{\otimes d} \\ \downarrow & & \downarrow \\ \mathrm{Alt}^p(V) \otimes_K \mathrm{Alt}^q(V) & \xrightarrow{\#} & \mathrm{Alt}^d(V). \end{array}$$

Proof. Unwrapping the definitions, this is precisely equivalent to lemma 3.3. □

We now establish the analogue of the well-known addition formula for exterior powers of vector spaces:

Proposition 3.5. *Let U and V be right A-modules. Then for any $d \in \mathbb{N}$ the shuffle product induces an isomorphism of $A^{\otimes d}$-modules :*

$$\bigoplus_{k=0}^{d} \mathrm{Alt}^k(U) \otimes_K \mathrm{Alt}^{d-k}(V) \xrightarrow{\sim} \mathrm{Alt}^d(U \oplus V).$$

Proof. Using the previous proposition, we easily establish that $\mathrm{Alt}^d(U \oplus V)$ is linearly spanned by the elements of the type $x_1 \# \cdots \# x_d$ with x_i in U or V. Now since the shuffle product is anti-symmetrical, we can permute the x_i so that $x_1, \dots, x_k \in U$ and $x_{k+1}, \dots, x_d \in V$. But any element of this type is obviously in the image of the map described in the statement of the proposition, so this map is surjective. We may then conclude that it is an isomorphism by checking the dimensions over K. □

Note that by construction, Alt^d is a covariant functor with respect to bimodule isomorphisms: if V, W are B-A-bimodules, and $f : V \to W$ is an isomorphism, there is a unique isomorphism $\mathrm{Alt}^d(f)$ of $B^{\otimes d}$-$A^{\otimes d}$-bimodules that makes this diagram commute:

$$\begin{array}{ccc} \mathrm{Alt}^d(V) & \xrightarrow{\mathrm{Alt}^d(f)} & \mathrm{Alt}^d(W) \\ {\scriptstyle j_V}\downarrow & & \downarrow{\scriptstyle j_W} \\ V^{\otimes d} & \xrightarrow{f} & W \end{array}$$

where j_V, j_W are the canonical inclusions.

4 Alternating powers of a ε-hermitian form

Now if V is a A-module equipped with a ε-hermitian form h with respect to some involution σ on A, we want to endow $\mathrm{Alt}^d(V)$ with an induced form $\mathrm{Alt}^d(h)$ such that in the split case we recover the exterior power of the bilinear form.

Lemma 4.1. *Let A be a central simple algebra over K. Then the element $s_{d,A} \in A^{\otimes d}$ is symmetric for the involution $\sigma^{\otimes d}$ for any involution σ on A.*

Proof. It follows directly from the fact, proved in lemma **??**, that the Goldman element is symmetric for $\sigma \otimes \sigma$. □

This observation allows the following definition:

Definition 4.2. *Let (A,σ) be an algebra with involution over K, and let (V,h) be a ε-hermitian module over (A,σ), with $B=\mathrm{End}_A(V)$. We set:*

$$\begin{array}{rccl} \mathrm{Alt}^d(h): & \mathrm{Alt}^d(V)\times \mathrm{Alt}^d(V) & \longrightarrow & A^{\otimes d} \\ & (s_{d,B}x, s_{d,B}y) & \longmapsto & h^{\otimes d}(x, s_{d,B}y) = h^{\otimes d}(s_{d,B}x, y). \end{array}$$

The equality on the right is a consequence of the fact that $s_{d,B}$ is symmetric for $\tau^{\otimes d}$ where τ is the adjoint involution of h. The left-hand side of the equality shows that the map is well-defined in $s_{d,B}y$, and the right-hand side shows that it is well-defined in $s_{d,B}x$.

Proposition 4.3. *The application $\mathrm{Alt}^d(h)$ is a ε^d-hermitian form over $(A^{\otimes d},\sigma^{\otimes d})$.*

Proof. We have for all $x,y\in V^{\otimes d}$ and all $a,b\in A^{\otimes d}$:

$$\begin{aligned} \mathrm{Alt}^d(h)(s_{d,B}x\cdot a, s_{d,B}y\cdot b) &= h^{\otimes d}(xa, s_{d,B}yb) \\ &= \sigma^{\otimes d}(a)h^{\otimes d}(x, s_{d,B}y)b \\ &= \sigma^{\otimes d}(a)\,\mathrm{Alt}^d(h)(s_{d,B}x, s_{d,B}y)b \end{aligned}$$

and

$$\begin{aligned} \mathrm{Alt}^d(h)(s_{d,B}y, s_{d,B}x) &= h^{\otimes d}(y, s_{d,B}x) \\ &= \varepsilon^d\sigma^{\otimes d}(h^{\otimes d}(s_{d,B}x, y)) \\ &= \varepsilon^d\sigma^{\otimes d}(\mathrm{Alt}^d(h)(s_{d,B}x, s_{d,B}y)). \end{aligned}$$ □

Remark 4.4. Since we defined $\mathrm{Alt}^d(V)$ as a submodule of $V^{\otimes d}$, in addition to $\mathrm{Alt}^d(h)$ it is also naturally equipped with the restriction of $h^{\otimes d}$, and we may wonder what the link is between the two. Since $h^{\otimes d}(s_{d,B}x, s_{d,B}y) = h^{\otimes d}(x, s_{d,B}^2 y)$ and $s_{d,B}^2 = d!s_{d,B}$ (which is easy to see from the definition), we can conclude that

$$h^{\otimes d}_{|\,\mathrm{Alt}^d(V)} = \langle d!\rangle\,\mathrm{Alt}^d(h).$$

In particular, in arbitrary characteristic we cannot simply define $\mathrm{Alt}^d(h)$ in terms of the restriction of $h^{\otimes d}$.

We can then show the compatibility of this construction with the sum formula:

Proposition 4.5. *Let (A,σ) be an algebra with involution over K, and let (U,h) and (V,h') be ε-hermitian modules over (A,σ). The module isomorphism in proposition 3.5 induces an isometry*

$$\bigoplus_{k=0}^{d}\mathrm{Alt}^k(h)\otimes_K \mathrm{Alt}^{d-k}(h') \xrightarrow{\sim} \mathrm{Alt}^d(h\perp h').$$

Proof. We set $B_1=\mathrm{End}_A(U)$, $B_2=\mathrm{End}_A(V)$, and $B=\mathrm{End}_A(U\oplus V)$. Let $u,u'\in U^{\otimes k}$ and $v,v'\in V^{\otimes d-k}$. Then

$$\begin{aligned} &\mathrm{Alt}^d(h\perp h')((s_{k,B_1}u)\#(s_{d-k,B_2}v),(s_{k,B_1}u)\#(s_{d-k,B_2}v)) \\ =&\,\mathrm{Alt}^d(h\perp h')(s_{d,B}(u\otimes v), s_{d,B}(u'\otimes v')) \\ =&(h\perp h')^{\otimes d}(s_{d,B}(u\otimes v), u'\otimes v') \\ =&\sum_{\pi\in\mathfrak{S}_d}(-1)^\pi(h\perp h')^{\otimes d}(g_B(\pi)(u\otimes v), u'\otimes v'), \end{aligned}$$

where we used lemma 3.2 for the first equality. We want to show that

$$(h \perp h')^{\otimes d}(g_B(\pi)(u \otimes v), u' \otimes v')$$

is zero if $\pi \notin \mathfrak{S}_{k,d-k}$. But if $u = x_1 \otimes \cdots \otimes x_k$, $u' = y_1 \otimes \cdots \otimes y_k$, and $v = x_{k+1} \otimes \cdots \otimes x_d$, $v' = y_{k+1} \otimes \cdots \otimes y_d$, then using lemma **??**:

$$\begin{aligned}
&(h \perp h')^{\otimes d}(g_B(\pi)(u \otimes v), u' \otimes v') \\
&= (h \perp h')^{\otimes d}((x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(d)}) \cdot g_A(\pi), (y_1 \otimes \cdots \otimes y_d)) \\
&= \sigma^{\otimes d}(g_A(\pi)) \cdot (h \perp h')(x_{\pi^{-1}(1)}, y_1) \otimes \cdots \otimes (h \perp h')(x_{\pi^{-1}(d)}, y_d)
\end{aligned}$$

which is indeed zero if $\pi \notin \mathfrak{S}_{k,d-k}$ since at least one of the $(h \perp h')(x_{\pi^{-1}(i)}, y_i)$ will be zero. Hence:

$$\begin{aligned}
&\mathrm{Alt}^d(h \perp h')(s_{k,B_1}u \# s_{d-k,B_2}v, s_{k,B_1}u \# s_{d-k,B_2}v) \\
&= \sum_{\pi \in \mathfrak{S}_{k,d-k}} (-1)^{\pi} (h \perp h')^{\otimes d}(g_B(\pi)(u \otimes v), u' \otimes v') \\
&= \sum_{\pi_1 \in \mathfrak{S}_k} \sum_{\pi_2 \in \mathfrak{S}_{d-k}} (-1)^{\pi_1 \pi_2} (h \perp h')^{\otimes d}(g_{B_1}(\pi_1)u \otimes g_{B_2}(\pi_2)v), u' \otimes v') \\
&= h(s_{k,B_1}u, u') \otimes h'(s_{d-k,B_2}v, v').
\end{aligned}$$

□

Starting from some morphism $f : (B, \tau) \to (A, \sigma)$ in $\mathbf{Br}_h(K)$, we have defined another morphism $\mathrm{Alt}^d(f)$, in $\mathbf{Br}_h(K)'$ when d is large enough, with target $(A^{\otimes d}, \sigma^{\otimes d})$. It is natural to try to understand what is the source object of this morphism; in other words, we want to study $\mathrm{End}_{A^{\otimes d}}(\mathrm{Alt}^d(V))$ and $\sigma_{\mathrm{Alt}^d(h)}$ in terms of (V, h). First we take a look at the special case of identity morphisms in $\mathbf{Br}_h(K)$.

Definition 4.6. *Let (A, σ) be an algebra with inveolution over K, and let $d \in \mathbb{N}$. We write $\Lambda^d_\sigma = \mathrm{Alt}^d(\langle 1 \rangle_\sigma)$, where we recall that $\langle 1 \rangle_\sigma$ is the identity of (A, σ) in $\mathbf{Br}_h(K)$.*

Then $(\lambda^d(A), \sigma^{\wedge d})$ is the algebra with involution such that

$$\Lambda^d_\sigma : (\lambda^d(A), \sigma^{\wedge d}) \to (A^{\otimes d}, \sigma^{\otimes d})$$

is a morphism in $\mathbf{Br}_h(K)$.

Note that this definition agrees with the one given in [**?**], ours being a reformulation in the language of $\mathbf{Br}_h(K)$ (the main difference is that in [**?**] $\sigma^{\wedge d}$ is defined directly and not as the adjoint involution of some hermitian form). The algebra $\lambda^d(A)$ is actually well-defined with no reference to any involution, simply by $\lambda^d(A) = \mathrm{End}_{A^{\otimes d}}(\mathrm{Alt}^d(A))$ where A is seen as a tautological A-module. If $d > \deg(A)$, then $\Lambda^d(A)$ is the zero ring.

Proposition 4.7. *Let $f : (B, \tau) \to (A, \sigma)$ be a morphism in $\mathbf{Br}_h(K)$. Then for any $d \in \mathbb{N}$ the following diagram in $\mathbf{Br}_h(K)'$ commutes:*

$$\begin{array}{ccc}
(\lambda^d(B), \tau^{\wedge d}) & & \\
\big\downarrow \Lambda^d_\tau & \searrow^{\mathrm{Alt}^d(f)} & \\
(B^{\otimes d}, \tau^{\otimes d}) & \xrightarrow{f^{\otimes d}} & (A^{\otimes d}, \sigma^{\otimes d}).
\end{array}$$

Proof. Say f corresponds to the ε-hermitian module (V, h). By definition, $f^{\otimes d} \circ \Lambda^d_\tau$ then corresponds to $(s_{d,B}B^{\otimes d} \otimes_{B^{\otimes d}} V^{\otimes d}, h^{\otimes d} \circ \mathrm{Alt}^d(\langle 1\rangle_\tau))$, where $s_{d,B}B^{\otimes d} \otimes_{B^{\otimes d}} V^{\otimes d} \simeq s_{d,B}V^{\otimes d} = \mathrm{Alt}^d(V)$ and $h^{\otimes d} \circ \mathrm{Alt}^d(\langle 1\rangle_\tau)$ is

$$((s_{d,B}x) \otimes u, (s_{d,B}y) \otimes v) \mapsto h^{\otimes d}(u, \sigma^{\otimes d}(x)s_{d,B}yv)$$

which under the above identification (taking $x = y = 1$) is exactly $\mathrm{Alt}^d(h)$. $\square$

Corollary 4.8. *Let f and g be two morphisms in $\mathbf{Br}_h(K)$ such that $f \circ g$ exists. Then for any $d \in \mathbb{N}$, we have in $\mathbf{Br}_h(K)'$:*

$$f^{\otimes d} \circ \mathrm{Alt}^d(g) = \mathrm{Alt}^d(f \circ g).$$

Proof. Write $f : (B, \tau) \to (A, \sigma)$ and $g : (C, \theta) \to (B, \tau)$. The result follows from the fact that the following diagram in $\mathbf{Br}_h(K)'$ commutes, which is established by two applications of proposition 4.7:

$$\begin{array}{ccccc} (\lambda^d(C), \theta^{\wedge d}) & \xrightarrow{\mathrm{Alt}^d(g)} & (B^{\otimes d}, \tau^{\otimes d}) & & \\ \downarrow \Lambda^d_\theta & & & \searrow^{\mathrm{Alt}^d(f\circ g)} & \\ (C^{\otimes d}, \theta^{\otimes d}) & \xrightarrow[g^{\otimes d}]{} & (B^{\otimes d}, \tau^{\otimes d}) & \xrightarrow[f^{\otimes d}]{} & (A^{\otimes d}, \sigma^{\otimes d}). \end{array}$$

$\square$

5 Exterior powers of a ε-hermitian module

In the case of vector spaces and bilinear forms, $(\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$ gave an appropriate definition for an operation $\lambda^d : SW(K) \to SW(K)'$, but for a general A it simply defines a map $SW^\varepsilon(A, \sigma) \to SW^{\varepsilon^d}(A^{\otimes d}, \sigma^{\otimes d})'$. We can then get back to (A, σ) using the isomorphism $\varphi^{(d)}_{(A,\sigma)}$ in $\mathbf{Br}_h(K)$ (recall definition **??**). Precisely:

Definition 5.1. *Let (A, σ) be an algebra with involution over K, and let $(V, h) \in SW^\varepsilon(A, \sigma)$. We set*

$$(\Lambda^d(V), \lambda^d(h)) = \varphi^{(d)}_{(A,\sigma)} \circ (\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$$

in $\mathbf{Br}_h(K)'$. This defines a map $\lambda^d : SW^\varepsilon(A, \sigma) \to SW(K)'$ if d is even, and $\lambda^d : SW^\varepsilon(A, \sigma) \to SW^\varepsilon(A, \sigma)'$ if d is odd.

Remark 5.2. Note that $\Lambda^d(V)$ depends on σ, even though V and $\mathrm{Alt}^d(V)$ are defined with no reference to any involution. On the other hand, $\Lambda^d(V)$ does not depend on h. See proposition 5.13 for an illustration of this. Furthermore, if $d > \mathrm{rdim}_A(V)$, then $\Lambda^d(V) = 0$.

Remark 5.3. If $(A, \sigma) = (K, \mathrm{Id})$, then $(\Lambda^d(V), \lambda^d(h)) \simeq (\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$ as bilinear spaces, and they coincide with the classical definition of exterior powers, but we have to pay attention to the grading. If (V, h) is in the odd component, then we have to distinguish two cases. When d is odd, $(\Lambda^d(V), \lambda^d(h))$ and $(\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$ correspond to the same element in the odd component. On the other hand, when d is even, then $(\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$ is still in the odd component, while $(\Lambda^d(V), \lambda^d(h))$ corresponds to the copy of that element in the even component. See remark 5.9 for further comments.

We want to show that this gives the structure we wanted on $\widetilde{GW}(A,\sigma)$. Recall (see [?]) that a pre-λ-ring is a commutative ring R endowed with maps $\lambda^d : R \to R$ for all $d \in \mathbb{N}$ such that for all $x, y \in R$, $\lambda^0(x) = 1$, $\lambda^1(x) = x$, and

$$\lambda^d(x+y) = \sum_{k=0}^{d} \lambda^k(x)\lambda^{d-k}(y).$$

Any family $(\lambda^d)_d$ of functions $R \to R$ can be encoded as a function $\lambda_t : R \to R[[t]]$, with $\lambda_t(x) = \sum_{d\in\mathbb{N}} \lambda^d(x)t^d$.

If we define $\Lambda(R) = 1 + tR[[t]]$ as the set of formal power series with constant coefficient 1, then $\Lambda(R)$ is a multiplicative subgroup of $R[[t]]^*$, with a group morphism $\eta : (\Lambda(R), \cdot) \to (R, +)$ which sends a formal series to its degree 1 coefficient. Then $(\lambda^d)_{d\in\mathbb{N}}$ defines a pre-λ-ring structure iff λ_t is a group morphism with η as a section.

Example 5.4. The ring $\mathbb{Z}$ is a pre-λ-ring, with $\lambda_t(n) = (1+t)^n$.

A pre-λ-ring morphism is a ring morphism that commutes with the operations λ^d. We say that R is an *augmented* pre-λ-ring if it is equipped with a pre-λ-ring morphism $R \to \mathbb{Z}$.

Example 5.5. The canonical exterior powers $\lambda^d : SW(K) \to SW(K)$ extend to functions $GW(K) \to GW(K)$ that give $GW(K)$ a natural pre-λ-ring structure. The dimension map $\dim : GW(K) \to \mathbb{Z}$ makes $GW(K)$ an augmented pre-λ-ring.

If R is a graded ring over some abelian group G, then we say it is a *graded* pre-λ-ring if furthermore $\lambda^d(R_g) \subset R_{dg}$ for all $g \in G$ (writing G additively). A graded pre-λ-ring morphism is a pre-λ-ring morphism that is also a homogeneous map.

Example 5.6. The pre-λ-ring structure on $GW(K)$ extends to a $\mathbb{Z}/2\mathbb{Z}$-graded pre-λ-ring structure on $GW^{\pm}(K)$.

Example 5.7. If R is a G-graded pre-λ-ring and H is an abelian group, then the group ring $R[H]$ is naturally a $(G\times H)$-graded pre-λ-ring, setting $\lambda^d(x\cdot h) = \lambda^d(x)\cdot(dh)$ for all $x \in R$ and $h \in H$. Then the augmentation map $R[H] \to R$ is a morphism of G-graded pre-λ-rings.

If R is a G-graded pre-λ-ring, we say it is augmented if it has a graded pre-λ-ring $R \to \mathbb{Z}[G]$. Composing with the augmentation $\mathbb{Z}[G] \to \mathbb{Z}$ then gives a non-graded augmentation $R \to \mathbb{Z}$.

We can then state:

Proposition 5.8. *Let (A,σ) be a central simple algebra with involution of the first kind over K. The maps λ^d defined in 5.1 on $SW^\varepsilon(K)$ and $SW^\varepsilon(A,\sigma)$ extend uniquely to maps $\lambda^d : \widetilde{GW}(A,\sigma) \to \widetilde{GW}(A,\sigma)$ such that $\widetilde{GW}(A,\sigma)$ is a Γ-graded pre-λ-ring. Furthermore, the dimension map* $\mathrm{rdim} : \widetilde{GW}(A,\sigma) \to \mathbb{Z}[\Gamma]$ *(see **??**) is a graded augmentation.*

Proof. Definition 5.1 gives functions λ^d from each component of $\widetilde{SW}(A,\sigma)$ to $\widetilde{GW}(A,\sigma)$, so they give functions λ_t from $\widetilde{SW}(A,\sigma)$ to $\Lambda(\widetilde{GW}(A,\sigma))$. Proposition 4.5 exactly shows that they are semi-group morphisms, so by the direct sum property they define a unique semi-group morphism λ_t from $\widetilde{SW}(A,\sigma)$ to $\Lambda(\widetilde{GW}(A,\sigma))$. Now the universal property of Grothendieck groups shows that this extends uniquely to a group morphism from $\widetilde{GW}(A,\sigma)$ to $\Lambda(\widetilde{GW}(A,\sigma))$.

The fact that η is a section (or equivalently that λ^1 is the identity) is clear since on each component of $\widetilde{SW}(A,\sigma)$, λ^1 is defined as the identity. So we have a pre-λ-ring structure on $\widetilde{GW}(A,\sigma)$. It preserves the grading since on each component of $\widetilde{SW}(A,\sigma)$, λ^d takes values in $SW(K)'$ if d is even, and in the component itself if d is odd.

The fact that rdim is a graded augmentation amounts to showing that if (V,h) has reduced dimension r, then $\Lambda^d(V)$ has reduced dimension $\binom{r}{d}$, which is a direct consequence of proposition 2.2. □

Remark 5.9. By examples 5.6 and 5.7, $GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$ is a Γ-graded pre-λ-ring, and the observations in remark 5.3 can be reformulated as: the natural isomorphism between $\widetilde{GW}(K,\mathrm{Id})$ and $GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$ is an isomorphism of Γ-graded pre-λ-rings.

Theorem 5.10. *The functor $\widetilde{GW}$ defines a functor from $\mathbf{Br}_h(K)$ to the category of Γ-graded pre-λ-rings.*

Proof. Let $f:(B,\tau)\to(A,\sigma)$ be a morphism in $\mathbf{Br}_h(K)$. The only thing to check is that the λ-operations are compatible with the ring morphism f_*, and it is enough to check this on $SW^{\varepsilon}(B,\tau)$. But if $g\in SW^{\varepsilon}(B,\tau)$, then we have in $\mathbf{Br}_h(K)'$:

$$\begin{aligned} f\circ\lambda^d(g) &= f\circ\varphi^{(d)}_{(B,\tau)}\circ \mathrm{Alt}^d(g)\\ &= \varphi^{(d)}_{(A,\sigma)}\circ f^{\otimes d}\circ \mathrm{Alt}^d(g)\\ &= \varphi^{(d)}_{(A,\sigma)}\circ \mathrm{Alt}^d(f\circ g)\\ &= \lambda^d(f\circ g)\end{aligned}$$

using proposition **??** and corollary 4.8. □

Remark 5.11. If $f:(B,\tau)\to(A,\sigma)$ is a morphism in $\mathbf{Br}_h(K)$, then since f_* is compatible with the λ-operations, we have $\lambda^d(f)=f_*(\lambda^d(\langle 1\rangle_\tau))$. Thus to be able to compute the exterior powers of any ε-hermitian form, we just need to be able to do the computation in the special case of diagonal forms $\langle 1\rangle_\sigma$ for any involution σ.

Example 5.12. Consider the split case $A=\mathrm{End}_K(V)$. Then the involution σ on A is adjoint to some bilinear form b on V, which is defined up to a scalar factor. Then if d is even $\lambda^d(b)$ is well-defined, while if d is odd it is only defined up to this same factor. On the other hand, the element $x_d=\lambda^d(\langle 1\rangle_\sigma)\in\widetilde{GW}(A,\sigma)$ is obviously well-defined. We can understand the relation between the two situations through functoriality. If we see the choice of b as a choice of isomorphism f_* from $\widetilde{GW}(A,\sigma)$ to $\widetilde{GW}(K,\mathrm{Id})\simeq GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$, then for d even $f_*(x_d)=\lambda^d(b)$ (in the even component) does not depend on f since $x_d\in GW(K)$, and for d odd $f_*(x_d)=\lambda^d(b)$ (in the odd component) depends on f since $x_d\in GW^{\varepsilon}(A,\sigma)$.

It is easy to see that the decomposition

$$A = \mathrm{Sym}(A,\sigma) \oplus \mathrm{Skew}(A,\sigma)$$

is orthogonal for the bilinear form $T_{\sigma,a,b}$ (recall proposition **??**) for any ε-symmetric $a, b \in A^*$. We write $T^+_{\sigma,a,b}$ (resp. $T^-_{\sigma,a,b}$) for the restriction of $T_{\sigma,a,b}$ to $\mathrm{Sym}(A,\sigma)$ (resp. $\mathrm{Skew}(A,\sigma)$).

Proposition 5.13. *Let (A,σ) be an algebra with involution over K, and let $a \in A^*$ be ε-symmetric. If σ is orthogonal (resp. symplectic), then $\Lambda^2(A)$ is naturally identified with* $\mathrm{Skew}(A)$ *(resp.* $\mathrm{Sym}(A,\sigma)$*), and $\lambda^2(\langle a\rangle_\sigma)$ is isometric to $\langle \frac{1}{2}\rangle T^-_{\sigma,a,a}$ (resp. $\langle \frac{1}{2}\rangle T^+_{\sigma,a,a}$).*

Proof. The right $(A \otimes_K A)$-module $\mathrm{Alt}^2(A)$ is by definition $(1-g_A)\cdot A \otimes_K A$. So since $\varphi^{(2)}_{(A,\sigma)}$ is given by the left $(A \otimes_K A)$-module A (with twisted action), we have

$$\Lambda^2(A) = (1-g_A)\cdot A \subset A.$$

But we saw in lemma **??** that under the twisted action, g_A acts on A as σ if σ is orthogonal, and $-\sigma$ if σ is symplectic. So $\Lambda^2(A)$ is the subspace of A consisting of anti-symmetrized elements if σ is orthogonal, and of symmetrized elements if σ is symplectic. Since 2 in invertible in K, this means that $\Lambda^2(A) = \mathrm{Skew}(A,\sigma)$ if σ is orthogonal, and $\Lambda^2(A) = \mathrm{Sym}(A,\sigma)$ if σ is symplectic (see for instance the discussion in [**?**, 2.A]).

Since we know from remark 4.4 that the restriction of $\langle a\rangle^2_\sigma$ to $\Lambda^2(A)$ is $\langle 2\rangle\lambda^2(\langle a\rangle_\sigma)$, and from proposition **??** that $\langle a\rangle^2_\sigma = T_{\sigma,a,a}$, we may conclude. □

6 Norm and determinant

The construction of the exterior powers $\lambda^d(A)$ of an algebra A of degree n has a very interesting special case, namely when $d = n$. Indeed, in this case $\lambda^n(A)$ has degree 1, so it is canonically isomorphic to K. This means that $\mathrm{Alt}^n(A)$ gives an explicit Brauer-equivalence between $A^{\otimes n}$ and K, which is a possible way to prove without using cohomology that the Brauer group is a torsion group, and that the exponent of A divides its degree (it is used for instance in [**?**], and the idea is attributed to Tamagawa).

In the split case, this is how one defines the determinant of an endomorphism: if $f \in \mathrm{End}_K(V)$, it induces $\Lambda^n(f) \in \mathrm{End}_K(\Lambda^n(V))$, which is a homothety since $\Lambda^n(V)$ has dimension 1, and the corresponding scalar $\det(f) \in K$ is called the *determinant* of f. We can imitate this definition in the general case: let A be a central simple algebra over K and $d \in \mathbb{N}$; there is an obvious action of $\mathfrak{S}_d$ on $A^{\otimes d}$, and we write $\mathrm{Sym}^d(A) \subset A^{\otimes d}$ for the subalgebra of fixed points under this action. Then any $x \in \mathrm{Sym}^d(A)$ commutes with $s_{d,A}$, so the application $a \mapsto xa$ on $A^{\otimes d}$ stabilizes $\mathrm{Alt}^d(A) = s_{d,A}A^{\otimes d}$. This defines a canonical map

$$\mathrm{Sym}^d(A) \longrightarrow \lambda^d(A).$$

The composition with the natural map $A \to \mathrm{Sym}^d(A)$ given by $a \mapsto a \otimes \cdots \otimes a$ then defines a map

$$\lambda^d : A \longrightarrow \mathrm{Sym}^d(A) \longrightarrow \lambda^d(A).$$

Clearly, λ^d is compatible with scalar extension, and when $A = \mathrm{End}_K(V)$ is split, this is the usual map from $\mathrm{End}_K(V)$ to $\mathrm{End}_K(\Lambda^d(V))$ induced by the functoriality of exterior powers.

Proposition 6.1. *Let A be a central simple algebra over K, of degree n. Then the map $\lambda^n : A \to K$ is the reduced norm Nrd_A.*

Proof. Since the reduced norm is usually defined by descent, and since the maps λ^d are compatible with base change, we can check this when A is split. But then as we discussed above λ^n is the determinant map, which is the reduced norm in the split case. □

Remark 6.2. This may be taken as a definition of the reduced norm, which avoids the use of Galois descent. In more concrete terms, this definition can be rephrased as: $a^{\otimes n} s_{d,n} = \mathrm{Nrd}_A(a) s_{d,n}$.

There is also a classical notion of determinant for algebras with involution, which can be defined as a descent of the determinant of a bilinear form (a definition that avoids any splitting argument is given in [?, 7.2]). However, it is only defined for algebras of even degree; the reason is clear if we use the descent definition: when the degree n of A is odd, then A is already split, but the bilinear form corresponding to the involution is only well-defined up to a scalar factor, so its determinant is not well-defined (this problem does not exist in even degree since the determinant of quadratic forms of even dimension is a similitude invariant). We suggest a slightly more general definition that works in arbitrary degree.

Definition 6.3. *Let (A, σ) be a central simple algebra with involution of the first kind over K, and let (V, h) be a ε-hermitian module over (A, σ). Then the determinant of (V, h) (often called the determinant of h) is*

$$\det(V, h) = \det(h) = \lambda^n(V, h) \in \widetilde{GW}(A, \sigma),$$

with $n = \mathrm{rdim}_A(V)$. The determinant of (A, σ) is

$$\det(A, \sigma) = \det(\sigma) = \det(\langle 1 \rangle_\sigma) \in \widetilde{GW}(A, \sigma).$$

Remark 6.4. If σ is symplectic, then usually there is no particular notion of $\det(\sigma)$, and indeed with our definition we always have $\det(\sigma) = \langle 1 \rangle \in GW(K)$.

Remark 6.5. When the degree of A is even and σ is orthogonal, then with this definition $\det(\sigma) \in GW(K)$ is a 1-dimensional quadratic form, so it has the form $\langle d \rangle$ for some $d \in K^*$, and by construction the square class of d is the usual determinant of σ as defined in [?] (we can check this in the split case). We usually identify the two definitions in this case (so we identify a square class wth the 1-dimensional form it defines).

Remark 6.6. When $\deg(A) = n$ is odd, then A is split and σ is orthogonal, and $\det(\sigma) \in GW(A, \sigma)$. The choice of some bilinear form b such that $\sigma = \sigma_b$ gives an isomorphism $\widetilde{GW}(A, \sigma) \simeq GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$ that sends $\det(\sigma)$ to $\langle \det(b) \rangle$ in the odd component. So while $\det(\sigma)$ is well-defined in $\widetilde{GW}(A, \sigma)$, its interpretation as a square class indeed depends on the choice of some b, which is the classical obstruction.

In the classical theory of bilinear forms, the pairing

$$\Lambda^p(V) \otimes_K \Lambda^q(V) \longrightarrow \Lambda^{p+q}(V)$$

induces a sort of duality when $p+q=n=\dim_K(V)$. Indeed, since $\Lambda^n(V)$ has dimension 1, by choosing a basis vector in $\Lambda^n(V)$ we get $\Lambda^p(V) \simeq \Lambda^q(V)^\vee$. If we restrain from choosing a basis of $\Lambda^n(V)$, then we get an isomorphism

$$\Lambda^p(V) \simeq \Lambda^n(V) \otimes_K \Lambda^q(V)^\vee \simeq \Lambda^n(V) \otimes_K \Lambda^q(V^\vee),$$

using that $\Lambda^d(V)^\vee \simeq \Lambda^d(V^\vee)$ for any $d \in \mathbb{N}$. Now if V is equipped with a non-degenerated bilinear form b, then using the identification $V \simeq V^\vee$ given by b, the isomorphism gives an isometry

$$(\Lambda^p(V), \lambda^p(b)) \simeq (\Lambda^n(V), \lambda^n(b)) \otimes_K (\Lambda^q(V), \lambda^q(b)).$$

So in the end we get an equality $\lambda^p(b) = \det(b)\lambda^q(b)$ in $GW(K)$ (or possibly in $GW^\pm(K)$ is b is anti-symmetric). We want to generalize this duality formula in the non-split case. First we show a general lemma on modules:

Lemma 6.7. *Let A and B be K-algebras, and let U, V and W be right modules over, respectively, A, B and $A \otimes_K B$. Then $U \otimes_K B$ is a right module over $A \otimes_K B \otimes_K B^{op}$, where B has its standard structure of $(B \otimes_K B^{op})$-module (here on the right). Likewise,* $\mathrm{Hom}_K(V,W)$ *is a module over $A \otimes_K B \otimes_K B^{op}$ through the action of $A \otimes_K B$ on W and the action of B on V.*

There is a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{A\otimes_K B}(U \otimes_K V, W) & \xrightarrow{\sim} & \mathrm{Hom}_{A\otimes_K B \otimes_K B^{op}}(U \otimes_K B, \mathrm{Hom}_K(V,W)) \\ f & \longmapsto & (u \otimes b \mapsto (v \mapsto f(u \otimes vb)))\,. \end{array}$$

Proof. A simple calculation shows that all actions are indeed respected, and that the inverse is given by $g \mapsto (u \otimes v \mapsto g(u \otimes 1)(v))$. □

Now if A and B are central simple algebras over K and V is a Morita B-A-bimodule, recall that $V^\vee$ is a A-B-bimodule, and we write V^{-1} for the same module seen as a B^{op}-A^{op}-bimodule.

Lemma 6.8. *There is a natural isomorphism of A-B-bimodule $V^\vee \simeq$* $\mathrm{Hom}_K(V,K)$, *given by either*

$$\begin{array}{rcl} \mathrm{Hom}_A(V,A) & \xrightarrow{\sim} & \mathrm{Hom}_K(V,K) \\ f & \longmapsto & \mathrm{Trd}_A \circ f \end{array}$$

or

$$\begin{array}{rcl} \mathrm{Hom}_B(V,B) & \xrightarrow{\sim} & \mathrm{Hom}_K(V,K) \\ f & \longmapsto & \mathrm{Trd}_B \circ f \end{array}$$

Proof. It is an easy verification that the maps are well-defined and are bimodule morphisms. To see that they are bijective it suffices to check that they are injective, since the K-dimensions are the same. But if f is in the kernel, it means that for any $v \in V$, we have for all $a \in A$:

$$\mathrm{Trd}_A(f(v)a) = \mathrm{Trd}_A(f(va)) = 0,$$

so $f(v) = 0$ since the trace form is non-degenerated.

Recall that the canonical identification $\mathrm{Hom}_A(V, A) \simeq \mathrm{Hom}_B(V, B)$ is given by

$$f(x)y = xf'(y)$$

for all $x, y \in V$, where $f \in \mathrm{Hom}_B(V, B)$ corresponds to $f' \in \mathrm{Hom}_A(V, A)$. Then to establish that the two isomorphisms correspond to each other through this identification, we have to show that $\mathrm{Trd}_B(f(x)) = \mathrm{Trd}_A(f'(x))$ for all $x \in V$. It is enough to check this in the split case; then $A \simeq \mathrm{End}_K(U) \simeq U \otimes_K U^\vee$, $B \simeq \mathrm{End}_K(W) \simeq W \otimes_K W^\vee$, and $V \simeq W \otimes_K U^\vee$. The reduced trace of A is given by $(u \otimes \varphi \mapsto \varphi(u))$, and likewise for B. We have an identification $V^\vee \simeq U \otimes_K W^\vee$, such that if f corresponds to $u \otimes \varphi$, then for $x = w \otimes \psi \in V$, $f(x) = \psi(u)w \otimes \varphi$ and $f'(x) = \varphi(w)u \otimes \psi$. In the end:

$$\mathrm{Trd}_B(f(x)) = \psi(u)\varphi(w) = \mathrm{Trd}_A(f'(x)). \qquad \square$$

Proposition 6.9. *Let A be a central simple algebra over K, let V be a right A-module of reduced dimension n, and let $p, q \in \mathbb{N}$ such that $p + q = n$. There is a canonical isomorphism of $(A^{\otimes n} \otimes_K (A^{op})^{\otimes q})$-modules*

$$\Phi_{p,q}(V) : \mathrm{Alt}^p(V) \otimes_K A^{\otimes q} \xrightarrow{\sim} \mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V)^{-1}.$$

Proof. If we apply the correspondance of lemma 6.7 to the shuffle map

$$\mathrm{Alt}^p(V) \otimes_K \mathrm{Alt}^q(V) \longrightarrow \mathrm{Alt}^n(V),$$

we get a morphism of $(A^{\otimes n} \otimes_K (A^{op})^{\otimes q})$-modules

$$\mathrm{Alt}^p(V) \otimes_K A^{\otimes q} \longrightarrow \mathrm{Hom}_K(\mathrm{Alt}^q(V), \mathrm{Alt}^n(V)),$$

and applying lemma 6.8 we get a morphism

$$\mathrm{Alt}^p(V) \otimes_K A^{\otimes q} \longrightarrow \mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V)^{-1}.$$

To prove that this is an isomorphism, it is enough to check it in the split case. Using the same notations as in the proof of lemma 6.8, this becomes a morphism

$$\Lambda^p(W) \otimes_K U^{\otimes p} \otimes_K U^{\otimes q} \otimes_K (U^\vee)^{\otimes q} \longrightarrow \Lambda^n(W) \otimes_K U^{\otimes n} \otimes_K (\Lambda^q(W))^\vee \otimes_K (U^\vee)^{\otimes q},$$

and it is a lengthy but simple verification to see that this is the tensor product of the usual isomorphism

$$\Lambda^p(W) \xrightarrow{\sim} \Lambda^n(W) \otimes_K (\Lambda^q(W))^\vee$$

with $U^{\otimes n} \otimes_K (U^\vee)^{\otimes q}$. $\square$

Remark 6.10. The module isomorphism $\Phi_{p,q}(V)$ induces an isomorphism between the endomorphism algebras of either side: this gives a canonical isomorphism $\lambda^p(A) \simeq \lambda^q(A)^{op}$. This is the isomorphism alluded to in [**?**, exercise II.12]. In particular, when $n = 2m$, this defines an isomorphism $\lambda^m(A) \simeq \lambda^m(A)^{op}$ which corresponds to the so-called canonical involution on $\lambda^m(A)$ (see [**?**, §10.B]).

If there is an involution σ on A, then the right $(A^{op})^{\otimes q}$-module $\mathrm{Alt}^q(V)^{-1}$ corresponds to the right $A^{\otimes q}$-module $\mathrm{Alt}^q(V)^*$, and the right $A^{\otimes q} \otimes_K (A^{op})^{\otimes q}$-module $A^{\otimes q}$ corresponds to the right $A^{\otimes 2q}$-module $\overline{A^{\otimes q}}$, where we see $A^{\otimes q}$ as a left $A^{\otimes 2q}$-module through the twisted sandwich action. So $\Phi_{p,q}(V)$ becomes an isomorphism of right $A^{\otimes n+q}$-modules

$$\mathrm{Alt}^p(V) \otimes_K \overline{A^{\otimes q}} \xrightarrow{\sim} \mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V)^*.$$

If furthermore we have a ε-hermitian form h on V, then this induces an isomorphism

$$\mathrm{Alt}^p(V) \otimes_K \overline{A^{\otimes q}} \xrightarrow{\sim} \mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V). \tag{3}$$

In addition, we can consider the inverse of $\varphi^{(2q)}_{(A,\sigma)}$ in $\mathbf{Br}_h(K)$ (recall **??**):

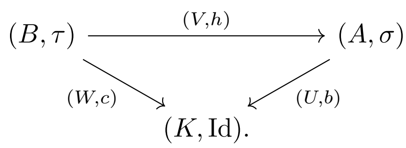
$$\overline{\varphi^{(2q)}_{(A,\sigma)}} : (K, \mathrm{Id}) \longrightarrow (A^{\otimes 2q}, \sigma^{\otimes 2q})$$

which is a hermitian form on $\overline{A^{\otimes q}}$. So each module in (3) carries a hermitian form.

Proposition 6.11. *Let (A,σ) be an algebra with involution over K, let (V,h) be a ε-hermitian module over (A,σ) of reduced dimension n, and let $p,q \in \mathbb{N}$ such that $p+q=n$. Then the $A^{\otimes n+q}$-module isomorphism (3) induced by $\Phi_{p,q}(V)$ is an isometry*

$$\mathrm{Alt}^p(h) \otimes \overline{\varphi^{(2q)}_{(A,\sigma)}} \xrightarrow{\sim} \mathrm{Alt}^n(h) \otimes \mathrm{Alt}^q(h).$$

Proof. To check that this gives an isometry, we can once again reduce to the split case, and we use the same notations as in the proof of proposition 6.9. We have bilinear forms b on U and c on W such that the following diagram in commutative in $\mathbf{Br}_h(K)$:



They induce identifications $U^\vee \simeq U$ and $W^\vee \simeq W$ given by $\hat{b}$ and $\hat{c}$, so in particular $A \simeq U \otimes_K U$ and $V \simeq W \otimes_K U$. Then the map (3) becomes

$$\Lambda^p(W) \otimes_K U^{\otimes p} \otimes_K U^{\otimes q} \otimes_K U^{\otimes q} \longrightarrow \Lambda^n(W) \otimes_K U^{\otimes n} \otimes_K \Lambda^q(W) \otimes_K U^{\otimes q},$$

and we can check that the hermitian forms which we have to show are isometric are, on the left:

$$(x \otimes u_1 \otimes u_2 \otimes u_3, y \otimes v_1 \otimes v_2 \otimes v_3) \mapsto \lambda^p(c)(x,y) \cdot (u_1 \otimes v_1) \otimes (u_2 \otimes v_2) \otimes (u_3 \otimes v_3)$$

with and on the right:

$$(x \otimes u \otimes y \otimes v, x' \otimes u' \otimes y' \otimes v') \mapsto \lambda^n(c)(x,x') \cdot \lambda^q(c)(y,y') \cdot (u \otimes u') \otimes (v \otimes v').$$

Thus this is the tensor product of the usual isometry

$$\lambda^p(c) \simeq \lambda^n(c)\lambda^q(c)$$

with $U^{\otimes n+q}$. □

We can finally prove:

Corollary 6.12. *Let (A,σ) be an algebra with involution over K, let (V,h) be a ε-hermitian module over (A,σ) of reduced dimension n, and let $p,q \in \mathbb{N}$ such that $p+q=n$. Then we have in $\widetilde{GW}(A,\sigma)$:*

$$\lambda^p(h) = \det(h) \cdot \lambda^q(h).$$

Proof. By definition, we have

$$\lambda^p(h) = \varphi_{(A,\sigma)}^{(n+q)} \circ \left(\mathrm{Alt}^p(h) \otimes \overline{\varphi_{(A,\sigma)}^{(2q)}}\right)$$

and

$$\det(h) \cdot \lambda^q(h) = \varphi_{(A,\sigma)}^{(n+q)} \circ \left(\mathrm{Alt}^n(h) \otimes \mathrm{Alt}^q(h)\right)$$

so this follows directly from the previous proposition. □