Lambda-ring properties of mixed Grothendieck-Witt rings of algebras of index 2

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Introduction

The ring structure of the Witt ring $W(K)$ of a field K (or its variant the Grothendieck-Witt ring $GW(K)$) is crucial to the study of quadratic forms over K. Given a central simple algebra with involution of the first kind (A, σ) over K, there is a strong interest in the literature about the Witt or Grothendieck-Witt groups $GW^\varepsilon(A, \sigma)$ (for $\varepsilon \in \mu_2(K)$), but the lack of ring structure on these groups limits the extension of some techniques used to study $GW(K)$. To address this issue, [2] introduced a graded commutative ring structure on

$$\widetilde{GW}(A, \sigma) = GW(K) \oplus GW^{-1}(K) \oplus GW^1(A, \sigma) \oplus GW^{-1}(A, \sigma).$$

In the same vein, in [3] the pre-λ-ring structure of $GW(K)$ was extended to a graded pre-λ-ring structure on $\widetilde{GW}(A, \sigma)$. One of the main goals was to produce Witt and cohomological invariants of ε-hermitian forms, and indeed this structure is crucially used in [4] to extend all "even" cohomological invariants of quadratic forms to anti-hermitian forms over quaternion algebras with their canonical involution.

Both from a purely structural perspective and in order to better control the invariants produced with those techniques, it is natural to try to understand the properties of the pre-λ-ring $\widetilde{GW}(A, \sigma)$, at least in some special cases. The first question that comes to mind is: is it actually a λ-ring? (In the older terminology: is the λ-ring $\widetilde{GW}(A, \sigma)$ special?) This seems reasonable since the corresponding property is true for $GW(K)$ (see [8]). We will prove that it is at least the case when $\mathrm{ind}(A) \leqslant 2$.

One of the classical constructions in the theory of λ-rings is that of the Adams operations ψ^d for $d \in \mathbb{N}^*$ (especially used in topological K-theory). The Adams operations on $GW(K)$ are characterized by the fact that ψ^d is the identity when d is odd, and ψ^2 is the dimension map. On $\widetilde{GW}(A, \sigma)$, we will see that the behaviour is more subtle: although ψ^d is still the identity when d is odd (Theorem 4.3), ψ^2 does not send a hermitian form to its (reduced) dimension but to a twisted version of it (see Theorem 4.4).

We develop in Section 3 the theory of those "twisted integers", and show that the previous observations are enough to give a conjectural formula for all ψ^d in the case of a general (A, σ), and to prove them when $\mathrm{ind}(A) \leqslant 2$. Finally, we study in Section 5 the question of products of λ-powers. If q is a quadratic form of dimension n, there is a formula to express $\lambda^i(q)\lambda^j(q)$ as a integer combination of the $\lambda^d(q)$, the coefficients depending only on n. We explain how to extend this, at least when $\mathrm{ind}(A) \leqslant 2$, by replacing integers with the "twisted integers" studied in Section 3 (see Theorem 5.7).

Notations and conventions

We fix a field K of characteristic not 2.

If (A, σ) is a central simple algebra with involution of the first kind over K, we will just say that (A, σ) is an algebra with involution over K. We define the sign $\varepsilon(\sigma)$ of the involution σ as $\varepsilon(\sigma) = 1$ when σ is orthogonal, and $\varepsilon(\sigma) = -1$ when σ is symplectic.

If $\varepsilon \in \mu_2(K)$, and (V, h) is a right ε-hermitian module over an algebra with involution (A, σ), and (B, τ) is its adjoint algebra with involution (so $B \simeq \mathrm{End}_A(V)$ and τ is the adjoint involution with respect to h), we say that (V, h) is an ε-hermitian Morita equivalence from (B, τ) to (A, σ).

We define Γ as the group $\mathbb{Z}/2\mathbb{Z} \times \mu_2(K)$.

Let G and H be commutative groups, R a G-graded ring (resp. abelian group) and S an H-graded ring (resp. abelian group). If $f : G \to H$ is a group morphism, an f-lax graded morphism $R \to S$ is a ring (resp. group) morphism $\varphi : R \to S$ such that for all $g \in G$, $\varphi(R_g) \subset S_{f(g)}$. This defines a category of graded rings (resp. abelian groups) with lax graded morphisms which are such pairs (f, φ).

1 Mixed Grothendieck-Witt rings

We recall here the construction and relevant properties of the mixed Grothendieck-Witt rings, taken from [2] and [3]. Let (A, σ) be an algebra with involution over K. Consider the Γ-graded abelian group

$$\widetilde{GW}(A, \sigma) = GW(K) \oplus GW^{-1}(K) \oplus GW^1(A, \sigma) \oplus GW^{-1}(A, \sigma).$$

It is functorial with respect to ε-hermitian Morita equivalences in the sense that if (V, h) is an ε-hermitian Morita equivalence from (B, τ) to (A, σ), it induces group isomorphisms

$$h_* : GW^{eps'}(B, \tau) \xrightarrow{\sim} GW^{\varepsilon\varepsilon'}(A, \sigma)$$

which, together with the identity of $GW^{\pm}(K)$, defines a group isomorphism $h_* : \widetilde{GW}(B, \tau) \xrightarrow{\sim} \widetilde{GW}(A, \sigma)$. It is an f_ε-lax graded isomorphism, with $f_\varepsilon : \Gamma \to \Gamma$ defined as $([i], \varepsilon') \mapsto ([i], \varepsilon\varepsilon')$.

Then there $\widetilde{GW}(A, \sigma)$ is naturally a Γ-structured ring. We refer to [3] for the details, but in summary this consists in:

- A commutative ring structure.
- Operations $\lambda^d : \widetilde{GW}(A, \sigma) \to \widetilde{GW}(A, \sigma)$ such that they form a pre-λ-ring structure on $\widetilde{GW}(A, \sigma)$, whose restriction to $GW^{\pm}(K)$ is its usual λ-ring structure, and such that $\lambda^d(GW^\varepsilon(A, \sigma)) \subset GW(K)$ if d is even, and $\lambda^d(GW^\varepsilon(A, \sigma)) \subset GW^\varepsilon(A, \sigma)$, when d is odd.

- A graded λ-morphism $\widetilde{\mathrm{rdim}} : \widetilde{GW}(A,\sigma) \to \mathbb{Z}[\Gamma]$ which is given componentwise by the usual reduced dimension map to $\mathbb{Z}$ (called an augmentation map).
- A positive structure, which is a rigid sub-λ-semiring $\widetilde{SW}(A,\sigma)$, which here simply consists of the classes of nonnegative reduced dimension.

They satisfy the following important properties

- The maps h_* induced by Morita equivalences preserve the λ-operations, and the augmentation map (and therefore the positive structure).
- The scalar extension maps $\widetilde{GW}(A,\sigma) \to \widetilde{GW}(A_L,\sigma_L)$ for any field extension L/K is a morphism of Γ-structured rings.
- In $\widetilde{GW}(A,\sigma)$, we have $\langle 1\rangle_\sigma^2 = T_\sigma \in GW(K)$, and $\lambda^2(\langle 1\rangle_\sigma) = \langle 2\rangle T_\sigma^{-\varepsilon(\sigma)}$.
- If $(A,\sigma) = (K,\mathrm{Id})$, $\widetilde{GW}(K,\mathrm{Id}) = GW^\pm(K) \oplus GW^\pm(K)$ is canonically isomorphic as a Γ-structured ring to the group algebra $GW^\pm(K)[\mathbb{Z}/2\mathbb{Z}]$. In practice, this means that if $x \in GW^\pm(K)$ is in the odd component, then $\lambda^d(x) \in \widetilde{GW}(K,\mathrm{Id})$ is the usual λ-power of x computed in $GW^\pm(K)$, but seen in the even component of $\widetilde{GW}(K,\mathrm{Id})$ if d is even, and in the odd component if d is odd.
- If $(A,\sigma) = (Q,\gamma)$ is a quaternion algebra with its canonical involution, then for any $z_1, z_2 \in Q_0^\times$,

  $$\langle z_1\rangle_\gamma \langle z_2\rangle_\gamma = \langle -\mathrm{Trd}_Q(z_1z_2)\rangle \varphi_{z_1,z_2}$$

  where φ_{z_1,z_2} is the 2-fold Pfister form whose Witt class is $\langle\langle z_1^2, z_2^2\rangle\rangle - n_Q$.
- If $h \in GW^\varepsilon(A,\sigma)$ has even reduced dimension $n > 0$, then $\lambda^n(h) = \langle \det(h)\rangle \in GW(K)$. Furthermore, for any $0 \leqslant d \leqslant n$, we have determinant duality: $\lambda^n(h) \cdot \lambda^d(h) = \lambda^{n-d}(h)$ ([5]).

2 The λ-ring property

One of the main questions that one can ask of the pre-λ-ring $\widetilde{GW}(A,\sigma)$ is: is it actually a λ-ring? Recall from [10] that a λ-ring is a pre-λ-ring R where $1 \in R$ is 1-dimensional and for each $x, y \in R$ we have identities

$$\lambda^n(xy) = P_n(\lambda^1(x), \dots, \lambda^n(x), \lambda^1(y), \dots, \lambda^n(y)) \tag{1}$$

and

$$\lambda^n(\lambda^m(x)) = P_{n,m}(\lambda^1(x), \dots, \lambda^{nm}(x)) \tag{2}$$

where $P_n \in \mathbb{Z}[x_1, \dots, x_n; y_1, \dots, y_n]$ and $P_{n,m} \in \mathbb{Z}[x_1, \dots, x_{nm}]$ are certain universal polynomials, which are homogeneous of respective degree $2n$ and nm if one gives x_i and y_i degree i.

Precisely, P_n is determined by the fact that if s_d (resp. σ_d) is the ith elementary symmetric polynomial in $x_1, \dots, x_n$ (resp. the $y_1, \dots, y_n$), so $s_d = \sum_{i_1 < \dots < i_d} x_{i_1} \cdots x_{i_d}$, then $P_n(s_1, \dots, s_n; \sigma_1, \dots, \sigma_n)$ is the coefficient of t^n in

$$\prod_{1 \leqslant i,j \leqslant n} (1 + x_i y_j t).$$

Likewise, $P_{n,m}$ is determined by the fact that if s_d is the elementary symmetric polynomial in $x_1, \dots, x_{nm}$, then $P_{n,m}(s_1, \dots, s_{nm})$ is the coefficient of t^n in

$$\prod_{1\leqslant i_1<\cdots<i_m\leqslant nm} (1 + x_i x_j t).$$

Rather than proving all those identities, one can use the following criterion, adapted from [11]:

Proposition 2.1. *Let G be an abelian group and R be a G-structured ring. If every homogeneous positive is a sum of line elements, then R is a λ-ring.*

Assume that every homogeneous positive element is a sum of positive elements of dimension at most 2, and that R is line-special, meaning that for any line element $\ell \in R$, any homogeneous positive element $x \in R$ and any $d \in \mathbb{N}$, one has $\lambda^d(\ell \cdot x) = \ell^d \cdot \lambda^d(x)$. Then R is a λ-ring when for every positive x and y of dimension 2, we have

(i) $\lambda^2(xy) = x^2\lambda^2(y) + \lambda^2(x)y^2 - 2\lambda^2(x)\lambda^2(y)$

(ii) $\lambda^3(xy) = xy\lambda^2(x)\lambda^2(y)$

(iii) $\lambda^4(xy) = (\lambda^2(x))^2(\lambda^2(y))^2$.

Proof. Since G is a group, the homogeneous positive structure $R_{\geqslant 0}$ is actually a positive structure on the (ungraded) pre-λ-ring R (this may fail when G is just a commutative monoid). Then the result is a combination of [11][3.1] and [11][3.4]. □

Remark 2.2. The restriction to G being a group is in no way crucial, but when G is a monoid one has to adapt the proof in [11] to take the grading into account.

Remark 2.3. It is easy to see that $\widetilde{GW}(A, \sigma)$ is line-special. If A is split, this claim can be checked in $GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$ where it follows from the familiar case of bilinear forms. If A is not split, then any line element has the form $\langle a\rangle \in GW(K)$ for some $a \in K^\times$, and it follows from the definition that $\lambda^d(\langle a\rangle h) = \langle a^d\rangle\lambda^d(h)$ for any ε-hermitian form h.

The first part of Proposition 2.1 is enough to prove that $GW(K)$ is a λ-ring, since every quadratic form can be diagonalized, which means that every positive element is a sum of line elements. This is indeed more or less how that fact is proved in [8].

But even for the slightly bigger $GW^{\pm}(K)$, one needs also the second part, since the positive elements in $GW^-(K)$, which are anti-symmetric forms, can only be written as multiples of the anti-symmetric hyperbolic plane $\mathcal{H}_{-1}$, of dimension 2.

The most general case we can prove as of now is:

Theorem 2.4. *If* $\mathrm{ind}(A) \leqslant 2$*, the pre-$\lambda$-ring $\widetilde{GW}(A, \sigma)$ is a λ-ring.*

Proof. The positive elements in $GW^{\pm}(K)$ are sums of line elements in $GW(K)$ and hyperbolic planes $\mathcal{H}_{-1}$ in $GW^{-1}(K)$.

If A is split, we may assume by hermitian Morita equivalence that $(A, \sigma) = (K, \mathrm{Id})$, and positive elements in the odd component of $\widetilde{GW}(K, \mathrm{Id})$ are again sums of line elements and $\mathcal{H}_{-1}$ (but seen in the odd component).

If A is not split, we can assume by hermitian Morita equivalence that (A,σ) is (Q,γ), a quaternion division algebra with its canonical involution. The positive elements in $GW^1(Q,\gamma)$ are sums of elements of the form $\langle a\rangle_\gamma$ with $a\in K^\times$, and those in $GW^{-1}(A,\sigma)$ of elements of the form $\langle z\rangle_\gamma$ with $z\in Q_0^\times$. Elements of both these forms are of dimension 2.

According to Proposition 2.1, we need to check equations (i), (ii) and (iii) when x and y are of the form $\mathcal{H}_{-1}$ (possibly seen in the odd component when $A=K$), $\langle a\rangle_\gamma$ (and we can reduce to $a=1$) and $\langle z\rangle_\gamma$.

Note that in all cases, xy is either hyperbolic (either because x or y is hyperbolic, or because $xy\in GW^{-1}(K)$) or a general 2-fold Pfister form (when x and y are in $GW^\varepsilon(Q,\gamma)$ for the same ε). So in all cases, $\det(xy)=1$. This takes care of equation (iii) since it can be rephrased as $\det(xy)=\det(x)^2\det(y)^2$. Also, because of determinant duality we have $\lambda^3(xy)=\det(xy)xy$, $\lambda^2(x)x=\det(x)x=x$ and likewise $\lambda^2(y)y=\det(y)y=y$, which shows equation (ii).

It remains to prove equation (i) when x and y are one of the three cases described above. Note that since in each case the dimensions coincide on both sides of the fomula, we may as well prove that the equality holds in $W(K)$ instead of $GW(K)$.

If x and y are both anti-symmetric planes, we use $x^2=y^2=xy=2\mathcal{H}$ (where $\mathcal{H}$ is the symmetric hyperbolic plane), and $\lambda^2(x)=\lambda^2(y)=\langle 1\rangle$.

We recall two formulas that are easy to establish, for any $a,b,c\in K^*$:

$$\lambda^2(\langle\langle a,b\rangle\rangle)=2(\langle\langle a,b\rangle\rangle-1) \tag{3}$$
$$\langle\langle a,c\rangle\rangle+\langle\langle b,c\rangle\rangle=\langle\langle ab,c\rangle\rangle+\langle\langle a,b,c\rangle\rangle\in W(K). \tag{4}$$

If $x=y=\langle 1\rangle_\gamma$, the equation is:

$$\lambda^2(\langle 2\rangle n_Q)=\langle 2\rangle n_Q+\langle 2\rangle n_Q-2$$

which follows from (3).

If $x=\mathcal{H}_{-1}$ and $y=\langle 1\rangle_\gamma$, then xy is hyperbolic. Let us take z_1,z_2 pure quaternions that anti-commute. Then $xy=\langle z_1,-z_1\rangle$, and $n_Q=\langle\langle z_1^2,z_2^2\rangle\rangle$. Equation (i) then becomes:

$$\lambda^2(\langle z_1,-z_1\rangle_\gamma)=2\mathcal{H}\cdot\langle 1\rangle+\langle 1\rangle\cdot\langle 2\rangle n_Q-2\langle 1\rangle\langle 1\rangle.$$

But

$$\begin{aligned}\lambda^2(\langle z_1,-z_1\rangle_\gamma)&=\lambda^2(\langle z_1\rangle_\gamma)+\langle z_1\rangle_\gamma\langle -z_1\rangle_\gamma+\lambda^2(\langle -z_1\rangle_\gamma)\\&=2\langle -z_1^2\rangle+\langle 2z_1^2\rangle\langle\langle z_1^2,-z_2^2\rangle\rangle\end{aligned}$$

so we can check that the equation can be rearranged as

$$\langle\langle -1,z_1^2\rangle\rangle=\langle 2\rangle\langle\langle z_1^2,z_2^2\rangle\rangle+\langle 2\rangle\langle\langle z_1^1,-z_2^2\rangle\rangle \tag{5}$$

which follows from (4).

If $x=\mathcal{H}_{-1}$ and $y=\langle z_1\rangle_\gamma$, then xy is hyperbolic, so $xy=\langle 1,-1\rangle_\gamma$. Let us choose some z_2 that anti-commutes with z_1. The equation then becomes:

$$\lambda^2(\langle 1,-1\rangle_\gamma)=2\mathcal{H}\cdot\langle -z_1^2\rangle+\langle -2z_1^2\rangle\langle\langle z_1^2,z_2^2\rangle\rangle-2\langle -z_1^2\rangle.$$

But since $\lambda^2(\langle 1,-1\rangle_\gamma)=2-\langle 2\rangle n_Q$, this can be rearranged to give the same equation as (5).

If $x = \langle z \rangle_\gamma$ and $y = \langle 1 \rangle_\gamma$, then taking some z_2 that anti-commutes with z, the equation becomes

$$\lambda^2(2\mathcal{H}_{-1}) = \langle -2z^2 \rangle \langle\langle z^2, z_2^2 \rangle\rangle + \langle\langle -2z^2 \rangle\rangle n_Q - 2\langle -z^2 \rangle.$$

But since $\lambda^2(2\mathcal{H}_{-1}) = 2(\mathcal{H} + 1)$, this can again be rearranged as (5).

Finally, if $x = \langle z_1 \rangle \gamma$ and $y = \langle z_2 \rangle \gamma$, then we choose z_0 that anti-commutes with both z_1 and z_2. The equation becomes:

$$\lambda^2(\langle -\operatorname{Trd}_Q(z_1 z_2) \rangle \varphi_{z_1,z_2}) = \langle 2z_1^2 z_2^2 \rangle \langle\langle z_1^2, -z_0^2 \rangle\rangle + \langle 2z_2^2 z_1^2 \rangle \langle\langle z_2^2, -z_0^2 \rangle\rangle - 2\langle -z_1^2 \rangle \langle -z_2^2 \rangle,$$

which using (3) gives:

$$2\varphi_{z_1,z_2} = \langle 2z_1^2 z_2^2 \rangle (\langle\langle z_1^2, -z_0^2 \rangle\rangle + \langle\langle z_2^2, -z_0^2 \rangle\rangle) + \langle\langle -1, z_1^2 z_2^2 \rangle\rangle.$$

Now $\varphi_{z_1,z_2} = \langle\langle z_1^2, z_2^2 z_0^2 \rangle\rangle = \langle\langle z_2^2, z_2^1 z_0^2 \rangle\rangle$ so in particular $2\varphi_{z_1,z_2}$ represents $2z_1^2 z_2^2$; also, $\langle\langle -1, z_1^2 z_2^2 \rangle\rangle$ represents $-2z_1^2 z_2^2$, so in the end we can rewrite the equation as:

$$\langle\langle -1, z_1^2, z_2^2 z_0^2 \rangle\rangle + \langle\langle -1, z_1^2 z_2^2 \rangle\rangle = \langle\langle z_1^2, -z_0^2 \rangle\rangle + \langle\langle z_2^2, -z_0^2 \rangle\rangle.$$

Using (4), this means:

$$\langle\langle -1, z_1^2, z_2^2 z_0^2 \rangle\rangle + \langle\langle -1, z_1^2 z_2^2 \rangle\rangle = \langle\langle -z_0^2, z_1^2 z_2^2 \rangle\rangle + \langle\langle -z_0^2, z_1^2, z_2^2 \rangle\rangle.$$

We can check using $\langle\langle z_1^2, z_0^2 \rangle\rangle = \langle\langle z_2^2, z_0^2 \rangle\rangle$ that the 2-fold and 3-fold Pfister forms on either side are the same, which concludes. ☐

We conjecture that the result holds without hypothesis on the index of A.

3 Brauer-Witt integers

In this section we define "Brauer integers" and "Brauer-Witt integer", which will be useful to us to describe Adams operations in $\widetilde{GW}(A, \sigma)$.

Lemma 3.1. *For any abelian group G, the graded pre-λ-ring structure on $\mathbb{Z}[G]$ induced by the unique λ-ring structure of $\mathbb{Z}$ is also a λ-ring structure.*

Proof. In general, if R is a λ-ring, it is not difficult to prove that $R[G]$ is also a λ-ring. But in this case we may notice that $\mathbb{Z}[G]$ has the homogeneous positive structure $\mathbb{N}[G]$ and that positive elements are sums of line elements (namely $1 \cdot g \in \mathbb{N}[G]$ for all $g \in G$). ☐

Definition 3.2. *Let $\mathbb{Z}_{\operatorname{Br}(K)}$ be the subgroup of the group ring $\mathbb{Z}[\operatorname{Br}(K)]$ consisting in elements $\sum_{\alpha \in \operatorname{Br}(K)} n_\alpha \cdot \alpha$ such that for all $\alpha \in \operatorname{Br}(K)$, $\operatorname{ind}(\alpha)$ divides n_α. We write $\mathbb{N}_{\operatorname{Br}(K)}$ for the submonoid of elements such that the n_α are in $\mathbb{N}$.*

For any central simple algebra B over K, we write $\{B\}$ for the element $\deg(B) \cdot [B] \in \mathbb{N}_{\operatorname{Br}(K)}$.

We call the elements of $\mathbb{Z}_{\operatorname{Br}(K)}$ Brauer integers.

Note that any element of $\mathbb{N}_{\operatorname{Br}(K)}$ can be uniquely written as $\sum_{i \in I} \{B_i\}$ where the B_i are central simple algebras from distinct Brauer classes.

Proposition 3.3. *If A and B are central simple algebras over K, and $d \in \mathbb{N}$ then in the λ-ring $\mathbb{Z}[\operatorname{Br}(K)]$ we have $\{A\} \cdot \{B\} = \{A \otimes_K B\}$, and $\lambda^d(\{A\}) = \{\lambda^d(A)\}$ if $d \leqslant \deg(A)$ (and $\lambda^d(\{A\}) = 0$ if $d > \deg(A)$).*

Proof. By definition, $\{A\} \cdot \{B\} = \deg(A)\deg(B) \cdot ([A]+[B])$, which is equal to $\deg(A \otimes_K B) \cdot [A \otimes_K B]$. Likewise, if $d \leqslant \deg(A)$, $\lambda^d(\{A\}) = \lambda^d(\deg(A)) \cdot d[A]$, which is $\binom{\deg(A)}{d} \cdot [A^{\otimes d}]$, and it is shown in [7] that $\deg(\lambda^d(A)) = \binom{\deg(A)}{d}$ and $[\lambda^d(A)] = [A^{\otimes d}]$. If $d > \deg(A)$ then $\binom{\deg(A)}{d} \cdot [A^{\otimes d}] = 0$. □

Corollary 3.4. *The* $\mathrm{Br}(K)$*-graded subgroup* $\mathbb{Z}_{\mathrm{Br}(K)}$ *of* $\mathbb{Z}[\mathrm{Br}(K)]$ *is actually a graded sub-*λ*-ring. It is a* $\mathrm{Br}(K)$*-structured ring, with positive structure* $\mathbb{N}_{\mathrm{Br}(K)}$*, and augmentation sending* $\{A\}$ *to* $\deg(A)$*.*

Proof. Since the products and λ-powers of homogeneous additive generators of $\mathbb{N}_{\mathrm{Br}(K)}$ are again in $\mathbb{N}_{\mathrm{Br}(K)}$, and $\mathbb{Z}_{\mathrm{Br}(K)}$ is additively generated by $\mathbb{N}_{\mathrm{Br}(K)}$, the result follows. □

Proposition 3.5. *There is a unique ring morphism*

$$\theta : \mathbb{Z}_{\mathrm{Br}(K)} \to GW(K)$$

such that if A *is a central simple algebra over* K *of degree* n*,* $\theta(\{A\})$ *is the unique element of* $GW(K)$ *of dimension* n *and Witt class* $[T_A] \in W(K)$*.*

Proof. Since T_A has dimension n^2, and $n^2 \equiv n \mod 2$, there is indeed a unique element $q \in GW(K)$ of dimension n and Witt class $[T_A]$. We send $\{A\}$ to that element, and that extends uniquely to a group morphism $\theta : \mathbb{Z}_{\mathrm{Br}(K)} \to GW(K)$.

Note that $\theta(\{K\}) = 1 \in GW(K)$ by construction since $T_K = 1$. A direct computation shows that $T_{A\otimes_K B} = T_A \otimes T_B$, so $\theta(\{A\} \cdot \{B\}) = \theta(\{A\})\theta(\{B\})$ and θ is a ring morphism. □

The image of θ is called the ring of Brauer-Witt integers of K.

Remark 3.6. The fact that $\mathrm{Trd}_A(xy) = \mathrm{Trd}_A(yx)$ shows that $T_A = T_{A^{op}}$. This is an indication that θ somehow only sees the Brauer classes mod 2.

We also consider a slight extension of the definition of $\mathbb{Z}_{\mathrm{Br}(K)}$ and θ: we define the ring of *signed* $\mathbb{Z}^{\pm}_{\mathrm{Br}(K)}$ Brauer integers as

$$\mathbb{Z}^{\pm}_{\mathrm{Br}(K)} = \mathbb{Z}_{\mathrm{Br}(K)}[\mu_2(K)]. \tag{6}$$

If $\varepsilon \in \mu_2(K)$, we write $\{A\}_\varepsilon$ for the element $\{A\} \cdot \varepsilon \in \mathbb{Z}_{\mathrm{Br}(K)}[\mu_2(K)]$. Then $\{A\}_1$ is identified with $\{A\}$ in the natural inclusion $\mathbb{Z}_{\mathrm{Br}(K)} \subset \mathbb{Z}^{\pm}_{\mathrm{Br}(K)}$.

Note that $\mathbb{Z}^{\pm}_{\mathrm{Br}(K)}$ is naturally a subring of $\mathbb{Z}[\mathrm{Br}(K) \times \mu_2]$, and it inherits its λ-ring structure. In particular,

$$\{A\}_\varepsilon \cdot \{B\}_{\varepsilon'} = \{A \otimes_K B\}_{\varepsilon\varepsilon'} \tag{7}$$

$$\lambda^d(\{A\}_\varepsilon) = \{\lambda^d(A)\}_{\varepsilon^d}. \tag{8}$$

We also extend θ to a morphism

$$\begin{array}{cccc} \theta^{\pm} : & \mathbb{Z}^{\pm}_{\mathrm{Br}(K)} & \longrightarrow & GW(K) \\ & \{A\}_\varepsilon & \longmapsto & \langle\varepsilon\rangle\theta(A). \end{array} \tag{9}$$

Its image is the ring of signed Brauer-Witt integers.

We introduce a last notation: let $\alpha \in \mathrm{Br}(K)$, $n \in \mathbb{N}$ divisible by $\mathrm{ind}(\alpha)$, and $\varepsilon \in \mu_2(K)$. Then

$$[n]_{\alpha,\varepsilon} = \theta^{\pm}(n \cdot (\alpha, \varepsilon)) = \theta^{\pm}(\{A\}_\varepsilon) \in GW(K) \tag{10}$$

where A is the central simple algebra with $\deg(A) = n$ and $[A] = \alpha$. We can summarize its properties: $\dim([n]_{\alpha,\varepsilon}) = n$, and its Witt class is $[n]_{\alpha,\varepsilon} = \varepsilon T_A \in W(K)$.

4 Adams operations

In any pre-λ-ring R, one may define the Adams operations $\psi^d : R \to R$ for all $d \in \mathbb{N}^*$. If we consider the generating series $\psi_t(x) : \sum_{d\in\mathbb{N}^*} \psi^d(x)t^d$, then we can define the ψ^d by setting

$$\psi_{-t}(x) = -t\frac{\partial_t(\lambda_t(x))}{\lambda_t(x)}. \tag{11}$$

One may check easily from that formula that for any $x \in R$:

$$x^2 = \psi^2(x) + 2\lambda^2(x) \tag{12}$$

Lemma 4.1. *Let R be a pre-λ-ring. Then for any $d \in \mathbb{N}^*$, ψ^d is an additive function on R such that for any 1-dimensional element $x \in R$ we have $\psi^d(x) = x^d$.*

Proof. Let $x, y \in R$. We have

$$\begin{aligned}
\psi_{-t}(x+y) &= -t\frac{\partial_t(\lambda_t(x+y))}{\lambda_t(x+y)} \\
&= -t\frac{\partial_t(\lambda_t(x)\lambda_t(y))}{\lambda_t(x)\lambda_t(y)} \\
&= -t(\frac{\partial_t(\lambda_t(x))}{\lambda_t(x)} + \frac{\partial_t(\lambda_t(y))}{\lambda_t(y)}) \\
&= \psi_{-t}(x) + \psi_{-t}(y).
\end{aligned}$$

If x has λ-dimension 1, then $\lambda_t(x) = 1 + xt$, so

$$\psi_t(x) = t\frac{x}{1-xt} = \sum_{d\in\mathbb{N}^*} x^d t^d.$$

□

Also note that if R is a G-graded pre-λ-ring, ψ^d is an operation of degree d, meaning that if $g \in G$ then

$$\psi^d(R_g) \subset R_{dg}. \tag{13}$$

Since the ψ^d are defined as certain integer coefficient polynomials of the λ^i, they commute with any λ-morphism. In particular, if the pre-λ-ring is augmented, the ψ^d preserve the augmentation.

We recall a crucial fact about Adams operations:

Proposition 4.2. *Let R be a λ-ring. Then for any $n, m \in \mathbb{N}^*$, ψ^n is an endomorphism of λ-ring of R, and we have $\psi^n \circ \psi^m = \psi^{nm}$.*

Proof. This is from [10]. $\square$

Theorem 4.3. *Let (A,σ) be an algebra with involution over K. For any odd $d \in \mathbb{N}^*$, ψ^d is the identity in $\widetilde{GW}(A,\sigma)$.*

Proof. Since d is odd and Γ is a 2-torsion group, ψ^d preserves the homogeneous components of $\widetilde{GW}(A,\sigma)$.

In $GW(K)$, we can write positive elements as sums of 1-dimensional forms $\langle a\rangle$, and $\psi^d(\langle a\rangle) = \langle a\rangle^d = \langle a\rangle$, so ψ^d is the identity on $GW(K)$.

If $x \in GW^{-1}(K)$, $\psi^d(x)$ is an element of $GW^{-1}(K)$ with dimension the same dimension as x (since ψ^d preserves the augmentation), so it must be x.

When A is split, the theorem is already proved, since after a Morita equivalence $\widetilde{GW}(A,\sigma) \simeq \widetilde{GW}(K,\mathrm{Id})$ (which commutes with ψ^d since it is a λ-morphism), we can apply the same reasoning to the odd component $GW^{\pm}(K)$.

Let $\varepsilon \in \mu_2(K)$ be such that $\varepsilon\varepsilon(\sigma) = 1$. Then from a theorem of Karpenko ([6]), if F is the generic splitting field of A (the function field of its Severi-Brauer variety), the scalar extension map

$$\widetilde{GW}(A,\sigma) \to \widetilde{GW}(A_F,\sigma_F)$$

is injective on the component $GW^{\varepsilon}(A,\sigma)$. Since it is a λ-morphism, it commutes with ψ^d, and since A_F is split ψ^d is the identity on $\widetilde{GW}(A_F,\sigma_F)$, so it is the identity on $GW^{\varepsilon}(A,\sigma)$.

Let us now assume that $(A,\sigma) = (Q,\gamma)$ is a quaternion algebra with its canonical involution. We prove that ψ^d is the identity on $GW^1(Q,\gamma)$. Note that ψ^d is a ring endomorphism of $\widetilde{GW}(Q,\gamma)$ since $\widetilde{GW}(Q,\gamma)$ is a λ-ring by Theorem 2.4. We need to check that for any $a \in K^{\times}$, $\psi^d(\langle a\rangle_\gamma) = \langle a\rangle_\gamma$. Since $\psi^d(\langle a\rangle_\gamma) = \psi^d(\langle a\rangle)\langle 1\rangle_\gamma$ and $\psi^d(\langle a\rangle) = \langle a\rangle$, we can reduce to $a = 1$. Also, since ψ^d preserves the dimension, we only need to prove that this equality holds in $W^1(Q,\gamma)$. We show this by induction on d.

This is clear for ψ^1. If we assume that ψ^m is the identity for all odd $m \leqslant d$, then since $\widetilde{GW}(Q,\gamma)$ is a λ-ring by theorem 2.4 we know that $\psi^n \circ \psi^m = \psi^{nm}$ for all $n, m \in \mathbb{N}^*$, so in particular if $m \leqslant d+1$ can be written $m = 2^i r$ with r odd, we have $\psi^m = (\psi^2)^i$. This means that, in $W(K)$, $\psi^m(\langle 1\rangle_\gamma)$ is $-T_Q$ if $i = 1$, and 2 if $i > 1$. Now we need the auxiliary fact that

$$T_Q\langle 1\rangle_\gamma = -2\langle 1\rangle_\gamma \in W(Q,\gamma). \tag{14}$$

Note that $q\langle 1_\gamma\rangle \mapsto qn_Q \in W(K)$ is injective on $W(Q,\gamma)$ (see [9][Ch 10, Thm 1.7]), so we need to show $T_Q n_Q = 2n_Q$ in $W(K)$. This can be done by direct computation: if $Q = (a,b)_K$ then $T_Q = \langle 2\rangle\langle 1, a, b, -ab\rangle$ and $n_Q = \langle\langle a, b\rangle\rangle$. Since n_Q represents $-a$, $-b$ and ab,

$$T_Q n_Q = \langle 2\rangle\langle 1, -1, -1, -1\rangle n_Q = -\langle 2\rangle 2n_Q = -2n_Q$$

where we use $\langle 2\rangle 2 = 2$ (since $2 = \langle\langle -1\rangle\rangle$ represents 2). This means that for all even $m \leqslant d+1$, $\psi^m(\langle 1\rangle_\gamma)\langle 1\rangle_\gamma = 2\langle 1\rangle_\gamma$. We may conclude using Newton's induction formula (see [10, 3.10]), which since $\lambda^m(\langle 1\rangle_\gamma) = 0$ for all $m > 2$ gives us

$$\psi^{d+2}(\langle 1\rangle_\gamma) = \psi^{d+1}(\langle 1\rangle_\gamma)\langle 1\rangle_\gamma - \psi^d(\langle 1\rangle_\gamma)\lambda^2(\langle 1\rangle_\gamma).$$

It just remains to notice that $\lambda^2(\langle 1\rangle_\gamma) = 1$, and use the induction hypothesis.

Finally, if A is arbitrary and $\varepsilon\varepsilon(\sigma) = -1$, we use another theorem of Karpenko ([6]) which implies that if F is the generic field of reduction to index 2 of A, namely the function field of the generalized Severi-Brauer variety $SB_2(A)$, then the scalar extension map

$$\widetilde{GW}(A,\sigma) \to \widetilde{GW}(A_F,\sigma_F)$$

is injective on $GW^\varepsilon(A,\sigma)$. Then we can check that ψ^d is the identity in $GW^\varepsilon(A,\sigma)$ by establishing it in $GW^\varepsilon(A_F,\sigma_F)$, where it follows from the previous case after a Morita equivalence from (A_F,σ_F) to some (Q,γ) over F. □

We can also compute ψ^2. First notice that there is a natural group embedding

$$\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mu_2(K) \simeq \langle [A]\rangle \times \mu_2(K) \hookrightarrow \mathrm{Br}(K) \times \mu_2(K). \tag{15}$$

This induces in turn a λ-ring embedding

$$\mathbb{Z}[\Gamma] \hookrightarrow \mathbb{Z}[\mathrm{Br}(K) \times \mu_2(K)] \tag{16}$$

and it is straightforward to check that this takes image in $\mathbb{Z}^\pm_{\mathrm{Br}(K)} \subset \mathbb{Z}[\mathrm{Br}(K) \times \mu_2(K)]$, which yields an embedding

$$\mathbb{Z}[\Gamma] \hookrightarrow \mathbb{Z}^\pm_{\mathrm{Br}(K)}. \tag{17}$$

Also note that since Γ is a group of exponent 2, for any even $d \in \mathbb{N}^*$, $\psi^2(\widetilde{GW}(A,\sigma)) \subset GW(K)$.

Theorem 4.4. *Let (A,σ) be an algebra with involution over K. Then $\psi^2 : \widetilde{GW}(A,\sigma) \to GW(K)$ is the composition*

$$\widetilde{GW}(A,\sigma) \xrightarrow{\widetilde{\mathrm{rdim}}} \mathbb{Z}[\Gamma] \hookrightarrow \mathbb{Z}^\pm_{\mathrm{Br}(K)} \xrightarrow{\theta^\pm} GW(K).$$

This means that if $x \in GW^\pm(K)$ then

$$\psi^2(x) = \dim(x) \in \mathbb{Z} \subset GW(K)$$

and if h is an ε-hermitian form over (A,σ) then

$$\psi^2(h) = [\mathrm{rdim}(h)]_{[A],\varepsilon\varepsilon(\sigma)}.$$

Proof. If $x \in GW^\pm(K)$, using the fact that ψ^2 is additive, we can reduce to the case where $x = \langle a\rangle$ or $x = \mathcal{H}_{-1}$. We have $\psi^2(\langle a\rangle) = \langle a\rangle^2 = 1$ by 4.1, and we use (12) to compute $\psi^2(\mathcal{H}_{-1})$:

$$\begin{aligned}\mathcal{H}_{-1}^2 &= 2\mathcal{H}_1 \\ &= 2 + 2\langle -1\rangle \\ &= \dim(\mathcal{H}_{-1}) + 2\lambda^2(\mathcal{H}_{-1}).\end{aligned}$$

Now assume that x is a positive element in $GW^\varepsilon(A,\sigma)$. Up to Morita equivalence, we may assume that $x = \langle 1\rangle_\sigma$. Then given the definition of $\theta^\pm$ we must show that

$$\langle 1\rangle_\sigma^2 = \varepsilon(\sigma) T_A + 2\lambda^2(\langle 1\rangle_\sigma) \tag{18}$$

in $W(K)$ (indeed the dimensions will coincide by definition of $\theta^{\pm}$ so it is enough to check equality in $W(K)$). This equality now follows from the fact that $\langle 1\rangle_\sigma^2 = T_\sigma$ and $\lambda^2(\langle 1\rangle_\sigma) = T_\sigma^{-\varepsilon(\sigma)}$, coupled with the familiar formulas $T_\sigma = T_\sigma^+ + T_\sigma^-$ and $T_A = T_\sigma^+ + \langle -1\rangle T_\sigma^-$. □

Corollary 4.5. *Let (A,σ) be an algebra with involution over K such that $\widetilde{GW}(A,\sigma)$ is a λ-ring (by Theorem 2.4 this includes the case where $\mathrm{ind}(A) \leqslant 2$), and let $d \in \mathbb{N}^*$.*

- *If d is odd, ψ^d is the identity.*
- *If $d \equiv 2 \mod 4$, ψ^d is*

$$\widetilde{GW}(A,\sigma) \xrightarrow{\widetilde{\mathrm{rdim}}} \mathbb{Z}[\Gamma] \hookrightarrow \mathbb{Z}^{\pm}_{\mathrm{Br}(K)} \xrightarrow{\theta^{\pm}} GW(K)$$

- *If $d \equiv 0 \mod 4$, ψ^d is*

$$\widetilde{GW}(A,\sigma) \xrightarrow{\mathrm{rdim}} \mathbb{Z} \subset GW(K).$$

Proof. The statement for d odd is Theorem 4.3. If $\widetilde{GW}(A,\sigma)$ is a λ-ring, then when $d = 2m$, we have $\psi^d = \psi^m \circ \psi^2$. The result then follows from the description of ψ^2 in Theorem 4.4, and the fact that the restriction of ψ^m to $GW(K)$ is the identity when m is odd and the dimension map when m is even (this follows from Theorems 4.4 and 4.3, but also directly from Lemma 4.1). □

Remark 4.6. Note that for an arbitrary (A,σ), the ψ^p for p prime are commuting ring endomorphisms of $\widetilde{GW}(A,\sigma)$ (since unless $p = 2$ they are the identity, and ψ^2 is easily seen to be a ring endomorphism). Since $\widetilde{GW}(A,\sigma)$ has no odd torsion as an abelian group, when p is odd ψ^p is a lift of the Frobenius modulo p. It is easy to see from formula (12) that ψ^2 is also a lift of the Frobenius modulo 2 (actually this holds for any p in any pre-λ-ring).

All this means that the ψ^p for prime p define a structure of special ψ-ring on $\widetilde{GW}(A,\sigma)$ in the sense of [10].

This is a good hint that $\widetilde{GW}(A,\sigma)$ is a λ-ring, but since it is not torsion-free as an abelian group we may not conclude anything (even when it is torsion-free, which is quite constraining, more so than just having K formally real and Pythagorean, we could not immediately conclude that the λ-structure coming from the ψ^p and the one we defined are identical).

5 Products of λ-powers

In $GW(K)$ it is easy to see that for any $n \in \mathbb{N}^*$ there are coefficients $L_{i,j,d}^{(n)} \in \mathbb{Z}$ such that for any q is a quadratic form of dimension n and any $i,j \in \mathbb{N}$,

$$\lambda^i(q)\lambda^j(q) = \sum_{d=0}^{i+j} L_{i,j,d}^{(n)} \lambda^d(q). \tag{19}$$

Actually, we can prove this exercise in [1] (in a slightly more general version):

Proposition 5.1. *Let R be a pre-λ-ring, and let $x \in R$ be an element which is a sum of n elements of dimension 1 whose squares are all equal to 1. Then in $R[u, v]$:*

$$\lambda_u(x)\lambda_v(x) = (1 + uv)^n \lambda_{\frac{u+v}{1+uv}}(x).$$

Proof. Write $f(x) = \lambda_u(x)\lambda_v(x)$ and rewrite the right-hand side as

$$g(x) = \sum_{d=0}^{n} (1 + uv)^{n-d}(u + v)^d \lambda^d(x) = (1 + uv)^n \lambda_{\frac{u+v}{1+uv}(x)}.$$

Since $f(x+y) = f(x)f(y)$ and $g(x+y) = g(x)g(y)$, we can reduce by hypothesis to the case where x has dimension 1 and $x^2 = 1$.

Then $f(x) = (1 + ux)(1 + vx) = 1 + (u + v)x + uv = g(x)$. □

Of course the hypothesis of the proposition is satisfied by a quadratic form of dimension n. We can rewrite the equality as

$$\lambda_u(x)\lambda_v(x) = \sum_{d=0}^{n} (1 + uv)^{n-d}(u + v)^d \lambda^d(x) \tag{20}$$

and extract from this:

$$L^{(n)}_{i,j,d} = \binom{n-d}{\frac{i+j-d}{2}} \binom{d}{\frac{d+i-j}{2}} \tag{21}$$

when $i + j \equiv d \mod 2$, and $L^{(n)}_{i,j,d} = 0$ otherwise.

The hypothesis on x in Proposition 5.1 is rather close to the fact that ψ^2 is the augmentation map and ψ^d is the identity when d is odd. In $\widetilde{GW}(A, \sigma)$ this is not quite exactly true, but ψ^2 is not that far from being the augmentation (ie dimension) map: it is a twisted version with values in Brauer-Witt integers instead of actual integers. This suggests that an analogue of (19) might still be true using Brauer-Witt integers for the $L^{(n)}_{i,j,d}$. We can at least show this in index 2.

Let us use for any $x \in \mathbb{Z}_{\mathrm{Br}(K)}$ the suggestive notation

$$(1 + t)^x := \theta(\lambda_t(x)) \in GW(K)[[t]]. \tag{22}$$

If $x \in \mathbb{N}_{\mathrm{Br}(K)}$ then $(1 + t)^x \in GW(K)[t]$. Of course when $x \in \mathbb{N} \subset \mathbb{N}_{\mathrm{Br}(K)}$ this is the usual polynomial.

Also, in any pre-λ-ring R, let us write

$$\lambda^{[0]}_t(x) = \sum_{d\in\mathbb{N}} \lambda^{2d}(x)t^d \tag{23}$$

$$\lambda^{[1]}_t(x) = \sum_{d\in\mathbb{N}} \lambda^{2d+1}(x)t^d, \tag{24}$$

so that

$$\lambda_t(x) = \lambda^{[0]}_{t^2}(x) + t\lambda^{[1]}_{t^2}(x). \tag{25}$$

Lemma 5.2. *Let R be a pre-λ-ring, and $x, y \in R$. Then*

$$\lambda^{[0]}_t(x + y) = \lambda^{[0]}_t(x)\lambda^{[0]}_t(y) + t\lambda^{[1]}_t(x)\lambda^{[1]}_t(y)$$

$$\lambda^{[1]}_t(x + y) = \lambda^{[0]}_t(x)\lambda^{[1]}_t(y) + \lambda^{[0]}_t(x)\lambda^{[1]}_t(y)$$

Proof. This is just an even/odd decomposition of the formula $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$. □

Lemma 5.3. *Let R be a pre-λ-ring, and $f, g \in R[[u,v]]$ be without constant terms. For every $x \in R$, we set*

$$\omega(x) = \lambda_f^{[0]}(x) + g\lambda_f^{[1]}(x).$$

Let $X \subset R$ be such that for every $x \in X$ and every odd $d \in \mathbb{N}$, $g^2\lambda^d(x) = f\lambda^d(x)$ in $R[[u,v]]$. Then for any $x, y \in X$,

$$\omega(x+y) = \omega(x)\omega(y).$$

Proof. We have

$$\begin{aligned}\omega(x+y) &= \lambda_f^{[0]}(x)\lambda_f^{[0]}(y) + f\lambda_f^{[1]}(x)\lambda_f^{[1]}(y)\\ &+ g\lambda_f^{[0]}(x)\lambda_f^{[1]}(y) + g\lambda_f^{[0]}(x)\lambda_t^{[f]}(y)\\ &= (\lambda_f^{[0]}(x) + g\lambda_f^{[1]}(x))(\lambda_f^{[0]}(y) + g\lambda_f^{[1]}(y))\end{aligned}$$

since $f\lambda_f^{[1]}(x)\lambda_f^{[1]}(y) = g^2\lambda_f^{[1]}(x)\lambda_f^{[1]}(y)$ by hypothesis. □

Remark 5.4. Note that given the hypothesis, if $x \in X$ then $\lambda_f^{[1]}(x) = \lambda_{g^2}^{[1]}(x)$.

Lemma 5.5. *Let Q be a quaternion algebra, and let $x \in GW^{-1}(Q,\gamma)$. Then for any even $n \in \mathbb{Z}$ we have*

$$[n]_{[Q]} \cdot x = n \cdot x \in GW(Q,\gamma).$$

In particular, $(1+t)^{mQ}x = (1+t)^{2m}x$ for any $m \in \mathbb{N}$.

Lemma 5.6. *Let F be the generic splitting field of Q. We have already mentioned that a theorem of Karpenko shows that $[n]_{[Q]} \cdot x = n \cdot x$ can be checked in $GW^{-1}(Q_F,\gamma_F)$ because the scalar extension map is injective on $GW^{-1}(Q,\gamma)$. But over Q_F is split so in $GW(F)$ we have $[n]_{[Q_F]} = n$.*

Since $(1+t)^{mQ} = \sum_{k=0}^{2m} \left[\binom{2m}{k}\right]_{k[Q]} t^k$, the last statement follows.

Then the analogue of Proposition 5.1 for algebras of index 2 is:

Theorem 5.7. *Let Q be a quaternion algebra, and let h be an anti-hermitian form over (Q,γ) of reduced dimension $n = 2m$. Then in $\widetilde{GW}(Q,\gamma)[u,v]$:*

$$\lambda_u(h)\lambda_v(h) = (1+uv)^{mQ}\left[\lambda^{[0]}_{\frac{(u+v)^2}{(1+uv)^Q}}(h) + \frac{u+v}{1+uv}\lambda^{[1]}_{\frac{(u+v)^2}{(1+uv)^Q}}(h)\right].$$

Proof. Let us write $f(h)$ for the left-hand side and $g(h)$ for the right-hand side of the equality. Clearly $f(h+h') = f(h)f(h')$. The fact that $g(h+h') = g(h)g(h')$ follows from Lemma 5.3 if we can show that

$$\left(\frac{u+v}{1+uv}\right)^2 \lambda^d(h) = \frac{(u+v)^2}{(1+uv)^Q}\lambda^d(h)$$

for every odd d. But in fact $\lambda^d(h) \in GW^{-1}(Q,\gamma)$, so we may apply Lemma 5.5.

So we are reduced to the case where $h = \langle z \rangle_\gamma$ for some $z \in Q_0^\times$. Then we need to show

$$\begin{aligned}&(1 + hu + \lambda^2(h)u^2)(1 + hv + \lambda^2(h)v^2)\\ =&(1 + uv)^Q + (1 + uv)(u + v)h + (u + v)^2\lambda^2(h).\end{aligned}$$

This amounts to

$$\begin{aligned}\lambda^2(h)h &= h\\ h^2 &= [2]_{[Q]} + 2\lambda^2(h)\\ (\lambda^2(h))^2 &= 1.\end{aligned}$$

The first formula is determinant duality since h has reduced dimension 2, the second one says that $\psi^2(h) = [2]_{[Q]}$ and the third one is because $\lambda^2(h)$ is a 1-dimensional quadratic form. □

Corollary 5.8. *With the same notations as in the theorem, we have for every* $0 \leqslant i, j \leqslant 2m$*:*

$$\lambda^i(h)\lambda^j(h) = \sum_{d=0}^{i+j} \widetilde{L}^{(n)}_{i,j,d}\lambda^d(h)$$

where

$$\widetilde{L}^{(n)}_{i,j,d} = \left[\binom{n-d}{\frac{i+j-d}{2}}\right]_{k[Q]} \binom{d}{\frac{d+i-j}{2}}$$

when $i + j \equiv d \mod 2$*, and* $\widetilde{L}^{(n)}_{i,j,d} = 0$ *otherwise. If d is odd, we can replace* $\widetilde{L}^{(n)}_{i,j,d}$ *by* $L^{(n)}_{i,j,d}$*.*

Proof. We can rewrite the formula in the theorem as

$$\lambda_u(h)\lambda_v(h) = \sum_{d \text{ odd}} (1 + uv)^{n-d}(u + v)^d\lambda^d(h) \tag{26}$$

$$+ \sum_{d \text{ even}} (1 + uv)^{\frac{n-d}{2}Q}(u + v)^d\lambda^d(h) \tag{27}$$

and then the rest is just a matter of identification of terms on both sides. The last statement follows from Remark 5.4. □

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