An Artin-Schreier-type theory for signatures of hermitian forms over involutions of the first kind

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Introduction

The theory of orderings of a field, as initiated by Artin and Schreier in [1], has strong ties with the structure of the Witt ring $W(K)$ of the field, through the study of signatures. Precisely, the set of orderings $X(K)$ of a field K correspond naturally to prime ideals of the Witt ring with residual characteristic 0, by assigning to each ordering the kernel of the corresponding signature map $W(K) \to \mathbb{Z}$ (see section **??**).

Various efforts have been made to extend this theory of signatures to involutions (in [7] for involutions of the first kind, and [8] for involutions of the second kind, see also [5, §11]) and hermitian forms, most notably by Astier and Unger in a series of articles (mainly [2] and [3]).

Unfortunately, the Witt group $W(A, \sigma)$ of an algebra with involution (A, σ) does not carry a natural ring structure, which prevents straightforward generalizations of the above result. This was somewhat circumvented in [3] where the authors introduce an ad hoc class of morphisms $W(A, \sigma) \to \mathbb{Z}$ which yields a similar classification result, but this is not fully satisfactory.

In [4], we introduced a ring structure $\widetilde{W}(A, \sigma)$ called the *mixed Witt ring* of an algebra with involution of the first kind (A, σ). Its construction and basic properties are briefly recalled in section **??**. The goal of this article is to show that the Artin-Schreier theory carries over if we consider this mixed Witt ring: the prime ideals of $\widetilde{W}(A, \sigma)$ of residual characteristic 0 are exactly the kernels of the signature maps $\widetilde{W}(A, \sigma) \to \mathbb{Z}$, and they are in a 2-to-1 correspondence with oderings of the base field (see **??**, where we give a description of the full prime spectrum of $\widetilde{W}(A, \sigma)$).

The fact that a given ordering of the base field corresponds to two signature maps has been a source of awkwardness in the previous literature, notably because this means that defining the total signature of a hermitian form as an appropriate map $X(K) \to \mathbb{Z}$ requires making some choices. Much of [2] and [3] can be interpreted as defining what constitutes a coherent choice, and showing that such coherent choices exist. We present an alternative approach: we define a completely canonical total signature $\widetilde{\operatorname{sign}}(x)$ which is a continuous function from $\widetilde{X}(A, \sigma)$ to $\mathbb{Z}$, where $\widetilde{X}(A, \sigma)$ is a canonical double-cover of $X(K)$, and only then do we investigate how to make pertinent choices to obtain a (non-canonical) total signature defined on $X(K)$, through the choice of a section of this double-cover (see **??**).

It should be noted that this article mainly provides a new language and a new framework for thinking about signatures of hermitian forms; the technical results are still heavily reliant on the work of Astier and Unger.

Conventions and notations

In all that follows, K is a field of characteristic not 2.

Its Witt ring is $W(K)$, and diagonal forms are denoted $\langle a_1, \dots, a_n \rangle$.

An algebra with involution (A, σ) over K is a central simple K-algebra of finite dimension, and σ is an involution of the first kind, meaning that σ is the identity on K. Recall that σ can be either orthogonal or symplectic.

We write $W^\varepsilon(A, \sigma)$ for the Witt group of regular ε-hermitian forms over (A, σ), with $\varepsilon = \pm 1$. If σ is orthogonal then we write $W_\varepsilon(A, \sigma) = W^\varepsilon(A, \sigma)$; if σ is symplectic then $W_\varepsilon(A, \sigma) = W^{-\varepsilon}(A, \sigma)$.

1 Orderings and signatures over fields

We start with a brief overview of the theory over fields, and we refer to [6] for proofs.

Recall that a field K is formally real if -1 is not a sum of squares in K; it is real closed if in addition no algebraic extension of K is formally real.

An ordering on a field K is a subgroup $P \subset K^*$ of index 2 which is stable under addition and does not contain -1. We write $X(K)$ for the set of all orderings of K. If $P \in X(K)$ we say that (K, P) is an ordered field, and we write $\mathrm{sign}_P : K^* \to \{\pm 1\}$ the morphism with kernel P. Then $\mathrm{sign}_P(a)$ is called the P-sign (or the sign if no confusion is possible) of $a \in K^*$. We can then speak of P-positive and P-negative elements.

An extension of an ordered field (K, P) is an ordered field (L, Q) such that L/K is an extension with $P = Q \cap K$. If L/K is algebraic and (L, Q) is real closed, then (L, Q) is called a real closure of K.

Proposition 1.1. *A field K admits an ordering iff it is formally real; a real closed field admits a unique ordering. Any ordered field (K, P) admits a real closure K_P, unique up to a unique K-isomorphism.*

Proposition 1.2. *If L is real closed, then there is a (unique) ring isomorphism between $W(L)$ and $\mathbb{Z}$, sending $\langle a \rangle$ to its sign relative to the unique ordering of L, for all $a \in L^*$.*

Thus for any ordering P on a field K, there is a unique ring morphism

$$\mathrm{sign}_P : W(K) \longrightarrow W(K_P) \xrightarrow{\sim} \mathbb{Z},$$

called the *signature* of K at P, which extends the P-sign map on K^* (meaning that $\mathrm{sign}_P(\langle a \rangle) = \mathrm{sign}_P(a)$). For any $p \in \mathbb{N}$ that is either 0 or a prime number, we write

$$\mathrm{sign}_{P,p} : W(K) \xrightarrow{\mathrm{sign}_P} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}.$$

Then we set

$$I_{P,p}(K) = \mathrm{Ker}(\mathrm{sign}_{P,p}).$$

We also write $I_P(K) = I_{P,0}(K)$.

Proposition 1.3. *For any ordering P on K, we have $I_{P,2}(K) = I(K)$. Furthermore,*

$$\mathrm{Spec}(W(K)) = \{I(K)\} \coprod_{P\in X(K)} \{I_{P,p}(K) \mid p \text{ odd or } 0\}.$$

In particular, there is a canononical identification between $X(K)$ and $\mathrm{Spec}_0(W(K))$.

Remark 1.4. If K is formally real, $X(K)$ is also identified with $\mathrm{minSpec}(W(K))$ (otherwise, $X(K)$ is empty and $\mathrm{minSpec}(W(K)) = \{I(K)\}$). This endows $X(K)$ with its so-called *Harrison topology.* In particular, $X(K)$ is Hausdorff and totally disconnected. In addition, it can be shown that it is compact, since the embedding $X(K) \to \{\pm 1\}^{K^*}$ given by $P \mapsto \mathrm{sign}_P$ is actually a closed immersion.

2 Hermitian Morita theory and the mixed Witt ring

In this section we review the necessary material from [4].

In [4], we define a category $\mathbf{Br}_h(K)$, called the hermitian Brauer 2-group of K, where the objects are the algebras with involutions (A,σ) over K, and the morphisms, which are all invertible, are the hermitian Morita equivalences, given by ε-hermitian modules (V,h), where V is a B-A-bimodule and $h : V\times V \to A$ is a ε-hermitian form such that the adjoint involution σ_h is equal to τ. There is a morphism $(B,\tau) \to (A,\sigma)$ in $\mathbf{Br}_h(K)$ iff A and B are Brauer-equivalent, and two choices of morphism are such that $h' = \langle\lambda\rangle h$ for some $\lambda \in K^*$.

Classical hermitian Morita theory (see [?]) yields that if $f : (B\tau) \to (A,\sigma)$ is a morphism in $\mathbf{Br}_h(K)$, then it induces natural $W(K)$-module isomorphisms $f_* : W_\varepsilon(B,\tau) \xrightarrow{\sim} W_\varepsilon(A,\sigma)$. In particular, if we consider the $W(K)$-modules

$$\widetilde{W_\varepsilon}(A,\sigma) = W(K) \oplus W_\varepsilon(A,\sigma)$$

and

$$\widetilde{W}(A,\sigma) = W(K) \oplus W_1(A,\sigma) \oplus W_{-1}(A,\sigma)$$

then $\widetilde{W}$ and $\widetilde{W_\varepsilon}$ are functors from $\mathbf{Br}_h(K)$ to the category of $W(K)$-modules.

In [4], we define a map

$$W_\varepsilon(A,\sigma) \times W_\varepsilon(A,\sigma) \to W(K)$$

which defines a commutative ring structure on $\widetilde{W}(A,\sigma)$ (with $\widetilde{W_\varepsilon}(A,\sigma)$ as a subring), such that the product of elements of $W_1(A,\sigma)$ and $W_{-1}(A,\sigma)$ is 0. The fundamental properties of this structure are:

- $\widetilde{W}$ is a functor from $\mathbf{Br}_h(K)$ to the category of commutative $W(K)$-algebras;
- if $h \in W_\varepsilon(A,\sigma)$, then $h^2 = T_{\sigma_h}$ where σ_h is the adjoint involution of h. In particular, $\langle 1\rangle_\sigma^2 = T_\sigma$.

When $(A,\sigma) = (K, \mathrm{Id})$, $\widetilde{W}(K, \mathrm{Id}) = W(K) \oplus W(K)$ is naturally isomorphic as a ring to the group algebra $W(K)[\mathbb{Z}/2\mathbb{Z}]$.

When $(A,\sigma) = (Q,\gamma)$ is a quaternion algebra with its canonical symplectic involution, we also have an explicit description of the product:

- for all $a, b \in K^\times$,
$$\langle a\rangle_\gamma \cdot \langle b\rangle_\gamma = \langle 2ab\rangle n_Q$$
where n_Q is the norm form of Q, ie the 2-fold Pfister form such that $e_2(n_Q) = [Q]$.
- for all pure quaternions $z_1, z_2 \in Q^\times$,
$$\langle z_1\rangle_\gamma \cdot \langle z_2\rangle_\gamma = \langle -\mathrm{Trd}_Q(z_1 z_2)\rangle \varphi_{z_1,z_2}$$
where φ_{z_1,z_2} is the 2-fold Pfister form such that $e_2(\varphi_{z_1,z_2}) = (z_1^2, z_2^2) + [Q]$.

3 Canonical retractions

The fundamental ingredient in the definition of signature maps on mixed Witt rings is the existence of certain natural ring morphisms:

Definition 3.1. *Let (A,σ) be an algebra with involution over K. We say that a ring morphism $\rho : \widetilde{W}(A,\sigma) \to W(K)$ is a retraction of $\widetilde{W}(A,\sigma)$ if it is the identity on $W(K)$. We define similarly retractions of $\widetilde{W}_\varepsilon(A,\sigma)$ which we also call orthogonal and symplectic retractions of $\widetilde{W}(A,\sigma)$ (depending on ε).*

Example 3.2. The augmentation map $W(K)[\mathbb{Z}/2\mathbb{Z}] \to W(K)$ defines a retraction ρ of $\widetilde{W}(K, \mathrm{Id})$, which we call the canonical retraction of $\widetilde{W}(K, \mathrm{Id})$.

We denote by $\mathbb{H}_K$ the Hamilton quaternion algebra over K, so that its Brauer class is $[\mathbb{H}_K] = (-1,-1) \in H^2(K,\mu_2)$. Recall that a classical result of Frobenius (at least for the real numbers) states that the only central division algebras over a real closed field are K and $\mathbb{H}_K$, which explains why the Hamilton quaternions play a crucial role in our exposition.

Proposition 3.3. *Let K be a real closed field. Then there is a (unique) retraction ρ of $\widetilde{W}(\mathbb{H}_K,\gamma)$, called the canonical retraction, such that $\rho(\langle 1\rangle_\gamma) = 2$ and $\rho(\langle z\rangle_\gamma) = 0$ for any non-zero pure quaternion $z \in \mathbb{H}_K$.*

Proof. The uniqueness of ρ is clear since $W^\pm(\mathbb{H}_K,\gamma)$ is generated as a $W(K)$-module by $\langle 1\rangle_\gamma$ and the $\langle z\rangle_\gamma$.

In general, for any quaternion algebra Q over any field K, if $(V,h) \in W(Q,\gamma)$, then there is a natural quadratic form $q_h : V \to K$ defined by $q_h(x) = h(x,x)$, and it is easy to see that if $h = \langle a_1,\dots,a_n\rangle_\gamma$ then $q_h = \langle a_1,\dots,a_n\rangle n_Q$. In our case, since $n_{\mathbb{H}_K} = \langle\langle -1,-1\rangle\rangle$, this means that $\langle a_1,\dots,a_n\rangle_\gamma \mapsto 4\langle a_1,\dots,a_n\rangle$ is a well defined $W(K)$-module morphism $W(\mathbb{H}_K,\gamma) \to W(K)$. Since $W(K)$ is torsion-free, we can divide by 2 and get a morphism sending $\langle 1\rangle_\gamma$ to 2. Thus there is a unique $W(K)$-module morphism ρ satisfying the conditions of the statement.

Clearly $\rho(xy) = \rho(x)\rho(y) = 0$ if $x \in W(\mathbb{H}_K,\gamma)$ and $y \in W^-(\mathbb{H}_K,\gamma)$. We just need to check that $\rho(\langle a\rangle_\gamma \cdot \langle b\rangle_\gamma) = \rho(\langle a\rangle_\gamma)\rho(\langle b\rangle_\gamma)$ for all $a, b \in K^*$, and

$\rho(\langle z_1\rangle_\gamma \cdot \langle z_2\rangle_\gamma) = \rho(\langle z_1\rangle_\gamma)\rho(\langle z_2\rangle_\gamma)$ for all $z_1, z_2 \in \mathbb{H}_K$ non-zero pure quaternions. According to proposition **??**, this respectively means that

$$\langle 2ab\rangle n_{\mathbb{H}_K} = (2\langle a\rangle) \cdot (2\langle b\rangle)$$

and

$$\langle -\operatorname{Trd}_Q(z_1 z_2)\rangle \varphi_{z_1,z_2} = 0$$

(unless z_1 and z_2 anti-commute, in which case the condition is trivial). The first one is true because $n_{\mathbb{H}_K} = 4 \in W(K)$ and $\langle 2\rangle$ is represented by $n_{\mathbb{H}_K}$; the second one is true because φ_{z_1,z_2} is hyperbolic. Indeed, the square of any pure quaternion in $\mathbb{H}_K$ is negative, so $(z_1^2, z_2^2) = (-1,-1) = [\mathbb{H}_K]$, which by definition of φ_{z_1,z_2} means it is hyperbolic. □

Remark 3.4. The proof shows that we can define a canonical retraction of $\widetilde{W}_{-1}(\mathbb{H}_K, \gamma)$ for any field K. Furthermore, we can define ρ on $\widetilde{W}(\mathbb{H}_K, \gamma)$ assuming only that K is Pythagorean. On the other hand, no retraction can exist on $\widetilde{W}(\mathbb{H}_K, \gamma)$ if the Pythagoras number of K is at least 3. We do not know whether there is always a retraction if the Pythagoras number is 2.

4 Signature maps

Since every central simple algebra over a real closed field is either split or Brauer-equivalent to the Hamilton quaternions, we can make the following definition:

Definition 4.1. *Let A be a central simple algebra over K. Then for any ordering $P \in X(K)$, we say that P is orthogonal with respect to A if A_{K_P} is split; otherwise, A_{K_P} is Brauer-equivalent to $\mathbb{H}_{K_P}$ and we say that P is symplectic with respect to A.*

The set of orthogonal orderings of K with respect to A is denoted $X_+(A)$, and the set of symplectic orderings is $X_-(A)$.

If $P \in X_+(A)$, we define $(D_P, \theta_P) = (K_P, \mathrm{Id})$; if $P \in X_-(A)$, then $(D_P, \theta_P) = (\mathbb{H}_{K_P}, \gamma)$.

Example 4.2. If A is split, $X_+(A) = X(K)$ and $X_-(A) = \emptyset$. On the other hand, $X_+(\mathbb{H}_K) = \emptyset$ and $X_-(\mathbb{H}_K) = X(K)$.

Remark 4.3. In the terminology of [2], $Nil[A,\sigma]$ is $X_+(A)$ if σ is symplectic, and $X_-(A)$ if σ is orthogonal. It is shown in [2, cor 6.5] that $X_+(A)$ and $X_-(A)$ are clopen in $X(K)$, and in particular are compact and totally disconnected (we will also provide a proof).

Let (A,σ) be an algebra with involution over K. For any $P \in X(K)$, let us choose an arbitrary isomorphism $f_P : (A_{K_P}, \sigma_{K_P}) \to (D_P, \theta_P)$ in $\mathbf{Br}_h(K_P)$. Then we define a ring morphism

$$\widetilde{\operatorname{sign}}_P^+ : \widetilde{W}(A,\sigma) \longrightarrow \widetilde{W}(A_{K_P}, \sigma_{K_P}) \xrightarrow{(f_P)_*} \widetilde{W}(D_P, \theta_P) \xrightarrow{\rho} W(K_P) \xrightarrow{\operatorname{sign}} \mathbb{Z}$$

and call it a *signature map* of (A,σ) at P. Any other choice of f_P has the form $\langle a\rangle f_P$ for some $a \in K_P^*$. Since K_P is real closed, $\langle a\rangle = \pm\langle 1\rangle$ in $W(K_P)$, so there are only two possible choices for f_P, and the other choice (when a is negative) leads to a different map $\widetilde{\operatorname{sign}}_P^-$; they are both equal to sign_P on $W(K)$, and they differ by a sign on $W^{\pm}(A,\sigma)$. The set $\{\widetilde{\mathrm{sign}}_P^+, \widetilde{\mathrm{sign}}_P^-\}$ is well-defined, but the labels $+$ and $-$ depend on the choice of f_P and are a priori arbitrary (but see section 6). When a statement does not depend on the choice of labels we will sometimes write $\widetilde{\mathrm{sign}}_P^{\pm}$ to designate either of the two possible maps.

Note that by construction of the canonical retractions ρ, $\widetilde{\mathrm{sign}}_P^{\pm}$ are both zero on $W_+(A,\sigma)$ if $P \in X_-(A)$ and are zero on $W_-(A,\sigma)$ if $P \in X_+(A)$. On the other hand, Astier and Unger show:

Lemma 4.4 ([2], thm 6.1)**.** *Let $\varepsilon = \pm 1$ and let $P \in X_\varepsilon(A)$. Then there exists $h \in W_\varepsilon(A,\sigma)$ such that $\widetilde{\mathrm{sign}}_P^{\pm}(h) \neq 0$. In particular, $\widetilde{\mathrm{sign}}_P^+$ and $\widetilde{\mathrm{sign}}_P^-$ are different functions on $\widetilde{W}(A,\sigma)$.*

We can now check that our construction is exhaustive:

Proposition 4.5. *Let (A,σ) be an algebra with involution over K, and let $\varepsilon, \varepsilon' = \pm 1$. For any $P \in X_\varepsilon(K)$, the only ring morphisms $\widetilde{W}_{\varepsilon'}(A,\sigma) \to \mathbb{Z}$ that extend the signature* sign_P *on $W(K)$ are $\widetilde{\mathrm{sign}}_P^+$ and $\widetilde{\mathrm{sign}}_P^-$ (with arbitrary labels $+$ and $-$).*

If $\varepsilon = \varepsilon'$ then these two morphisms are distinct on $\widetilde{W}_{\varepsilon'}(A,\sigma)$, while if $\varepsilon \neq \varepsilon'$ they coincide on $\widetilde{W}_{\varepsilon'}(A,\sigma)$ and are both zero on $W_{\varepsilon'}(A,\sigma)$.

Proof. Let $f : \widetilde{W}_{\varepsilon'}(A,\sigma) \to \mathbb{Z}$ be a ring morphism extending sign_P, and let $x \in W_{\varepsilon'}(A,\sigma)$. Then $f(x)^2 = \mathrm{sign}_P(x^2) = \widetilde{\mathrm{sign}}_P^{\pm}(x)^2$, so $f(x) = \widetilde{\mathrm{sign}}_P^{s(x)}(x)$ for some $s(x) = \pm 1$. Now we want to show that we can take $s(x)$ constant.

If $\varepsilon \neq \varepsilon'$, we already noticed that both $\widetilde{\mathrm{sign}}_P^{\pm}$ are zero on $W_{\varepsilon'}(A,\sigma)$ so we can choose $s(x)$ arbitrarily for all x.

If $\varepsilon = \varepsilon'$, according to lemma 4.4, there is some $y \in W_{\varepsilon'}(A,\sigma)$ such that $\widetilde{\mathrm{sign}}_P^{\pm}(y) \neq 0$; in particular, $s = s(y)$ is uniquely determined. Now for an arbitrary x, if $\widetilde{\mathrm{sign}}_P^{\pm}(x) = 0$ then we can choose $s(x)$ arbitrarily so we take $s(x) = s$, and if $\widetilde{\mathrm{sign}}_P^{\pm}(x) \neq 0$, then

$$f(xy) = \mathrm{sign}_P(xy) = \widetilde{\mathrm{sign}}_P^{s}(xy) = s(x)s \cdot \widetilde{\mathrm{sign}}_P^{s(x)}(x)\widetilde{\mathrm{sign}}_P^{s}(y) = s(x)s \cdot f(x)f(y)$$

and since $f(x)f(y) \neq 0$, we have $s(x)s = 1$. ☐

Remark 4.6. In [3, prop 7.4], lacking a a ring structure, Astier and Unger show a slightly stronger result (with the cost of a more involved proof): $\widetilde{\mathrm{sign}}_P^+$ is the only $W(K)$-module morphisms extending sign_P, where we see $\mathbb{Z}$ as a $W(K)$-module through sign_P, up to multiplication by an arbitrary integer on $W^{\pm}(A,\sigma)$. So the only thing that compatibility with the product of hermitian forms adds to our statement is a normalization condition (our morphisms can only differ by a sign on $W^{\pm}(A,\sigma)$ and not an arbitrary integer). On that subject, it should be noted that they normalize the signature maps at symplectic orderings so that they give surjective maps $W_-(A,\sigma) \to \mathbb{Z}$, while with our construction we get a map to $2\mathbb{Z}$ (which is necessary to get a ring morphism).

Corollary 4.7. *Let (A,σ) be an algebra with involution over K, and let $P \in X(K)$. There are exactly two different ring morphisms $\widetilde{W}(A,\sigma) \to \mathbb{Z}$ that extend the signature* sign_P *on $W(K)$, namely $\widetilde{\mathrm{sign}}_P^+$ and $\widetilde{\mathrm{sign}}_P^-$ (with arbitrary labels $+$ and $-$).*

Remark 4.8. We can also define a signature of the involution σ: $\mathrm{sign}^{\pm}_P(\sigma) \overset{\mathrm{def}}{=} \widetilde{\mathrm{sign}}^{\pm}_P(\langle 1\rangle_\sigma)$. We again encounter a sign ambiguity, which is why in [7] only the absolute value $|\,\mathrm{sign}^{\pm}_P(\sigma)|$ is defined, and taken as the definition of the signature of σ. Note that the definitions agree since they characterize $\mathrm{sign}_P(\sigma) \in \mathbb{N}$ by $\mathrm{sign}_P(\sigma)^2 = \mathrm{sign}_P(T_\sigma)$, and of course $T_\sigma = \langle 1\rangle_\sigma^2$ in $\widetilde{W}(A,\sigma)$.

5 The spectrum of the mixed Witt ring

Now that we have our signature maps, we want to obtain a description of $\mathrm{Spec}(\widetilde{W}(A,\sigma))$ similar to proposition 1.3 for $W(K)$.

Let $P \in X(K)$ and let $p \in \mathbb{N}$ be either 0 or a prime number. Assume we chose a labelling of $\widetilde{\mathrm{sign}}^{+}_P$ and $\widetilde{\mathrm{sign}}^{-}_P$. Then we define

$$\widetilde{\mathrm{sign}}^{\pm}_{P,p} : \widetilde{W}(A,\sigma) \xrightarrow{\widetilde{\mathrm{sign}}^{\pm}_P} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$$

and

$$I^{\pm}_{P,p}(A,\sigma) = \mathrm{Ker}(\widetilde{\mathrm{sign}}^{\pm}_{P,p}),$$

which is by construction a prime ideal of $\widetilde{W}(A,\sigma)$ (maximal if $p \neq 0$).

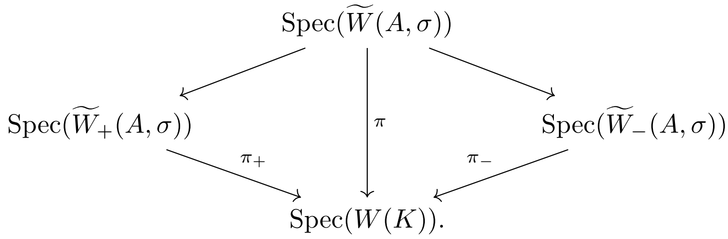
Let $P \in X_\varepsilon(K)$. Then since $\widetilde{\mathrm{sign}}^{\pm}_P$ is zero on $W_{-\varepsilon}(A,\sigma)$, we can write

$$I^{\pm}_{P,p}(A,\sigma) = J^{\pm}_{P,p}(A,\sigma) \oplus W_{-\varepsilon}(A,\sigma)$$

with

$$J^{\pm}_{P,p}(A,\sigma) = I^{\pm}_{P,p}(A,\sigma) \cap W_\varepsilon(A,\sigma).$$

Proposition 5.1. *Let (A,σ) be an algebra with involution over K. Then $I(A,\sigma)$ is the only prime ideal of $\widetilde{W}(A,\sigma)$ with residual characteristic 2; in particular, for any $P \in X(K)$, we have $I^{\pm}_{P,2}(A,\sigma) = I(A,\sigma)$. Furthermore, consider the following natural commutative diagram of schemes:*



The fiber of π above $I(K) \in \mathrm{Spec}(W(K))$ is $\{I(A,\sigma)\}$.

Let $P \in X(K)$, and p be either 0 or an odd prime. The fiber of π above $I_{P,p}(K)$ is $\{I^{+}_{P,p}(A,\sigma), I^{-}_{P,p}(A,\sigma)\}$ (the two being distinct).

If $P \in X_\varepsilon(A)$, the fiber of π_ε above $I_{P,p}(K)$ is $\{J^{+}_{P,p}(A,\sigma), J^{-}_{P,p}(A,\sigma)\}$ (the two being distinct), while the fiber of $\pi_{-\varepsilon}$ is $\{I_{P,p}(K) \oplus W_{-\varepsilon}(A,\sigma)\}$.

Proof. Let $I \subset \widetilde{W}(A,\sigma)$ be a prime ideal with residual characteristic 2. Then $I \cap W(K)$ is a prime ideal with residual characteristic 2, so $I \cap W(K) = I(K)$. If $x \in W^{\pm}(A,\sigma)$, then $x \in I$ iff $x^2 \in I(K)$, which is equivalent to $\mathrm{rdim}_2(x)^2 = 0 \in \mathbb{Z}/2\mathbb{Z}$, so $x \in I(A,\sigma)$. So $I(A,\sigma) \subset I$, and since $I(A,\sigma)$ is a maximal ideal we have equality. This implies the statement about $I^{\pm}_{P,2}(A,\sigma)$ and about the fiber of π above $I(K)$.

Now let $P \in X_\varepsilon(K)$, and p be either 0 or an odd prime; we set $R = \mathbb{Z}/p\mathbb{Z}$. Let I be in the fiber of $\pi_{\varepsilon'}$ above $I_{P,p}(K)$, and let $f : \widetilde{W}_{\varepsilon'}(A,\sigma) \to S$ be the surjective morphism with kernel I, with $R \subset S$. Then the same proof as for proposition 4.5 shows that $R = S$ and $f = \widetilde{\operatorname{sign}}^{s}_{P,p}$ for some $s = \pm 1$ (which is uniquely determined when $\varepsilon = \varepsilon'$). Indeed, we show the same way that for fixed $x \in W_{\varepsilon'}(A,\sigma)$ we have $f(x)^2 = \widetilde{\operatorname{sign}}^{\pm}_{P,p}(x)^2 \in R$, which shows that $R = S$ since S is integral. The rest of the reasoning is also the same, the only difference being that we have to invoque that $p \neq 2$ to justify that we get two different signature maps when $\varepsilon = \varepsilon'$.

Now suppose I is in the fiber of π above $I_{P,p}(K)$. Then we just showed that $I \cap \widetilde{W}_\varepsilon(A,\sigma) = J^s_{P,p}$ for some $s = \pm 1$ and that $I \cap \widetilde{W}_{-\varepsilon}(A,\sigma) = I_{P,p}(K) \oplus W_{-\varepsilon}(A,\sigma)$. This shows that $I = I^s_{P,p}(A,\sigma)$. □

Remark 5.2. In the continuity of remark 4.6, Astier and Unger show in [3, 6.5,6.7] slightly different and arguably stronger results, since they obtain a similar classification without asking that their "ideals" be stable by multiplication by a hermitian form. There is however a difference for primes above $I(K)$, since they find many such "ideals" (but of course only $I(A,\sigma)$ is an actual ideal).

Emulating the classical case, we set

$$\widetilde{X}(A,\sigma) = \operatorname{Spec}_0(\widetilde{W}(A,\sigma))$$

as a topological subspace of $\operatorname{Spec}(\widetilde{W}(A,\sigma))$; its elements are the $I^{\pm}_P$ for $P \in X(K)$. When K is formally real, this is also $\operatorname{minSpec}(\widetilde{W}(A,\sigma))$ (otherwise, $\widetilde{X}(A,\sigma)$ is empty, while $\operatorname{minSpec}(\widetilde{W}(A,\sigma))$ is a single point). Thus the continuous map $\pi : \operatorname{Spec}(\widetilde{W}(A,\sigma)) \to \operatorname{Spec}(W(K))$ induces a continuous two-to-one map $\overline{\pi} : \widetilde{X}(A,\sigma) \to X(K)$. We also set $\widetilde{X}_\varepsilon(A,\sigma) = \pi^{-1}(X_\varepsilon(A))$, so that $\widetilde{X}_\varepsilon(A,\sigma) \to X_\varepsilon(A)$ is also a continuous two-to-one map. We easily see from proposition 5.1 that $\operatorname{Spec}_0(\widetilde{W}_\varepsilon(A,\sigma))$ is canonically identified with $\widetilde{X}_\varepsilon(A,\sigma) \coprod X_{-\varepsilon}(K)$.

As in the classical case we have a total signature:

Definition 5.3. *Let (A,σ) be an algebra with involution over K. The total signature of any $x \in \widetilde{W}(A,\sigma)$ is the function*

$$\widetilde{\operatorname{sign}}(x) : \widetilde{X}(A,\sigma) \longrightarrow \mathbb{Z}$$

such that $\widetilde{\operatorname{sign}}(x)(I^\varepsilon_P) = \widetilde{\operatorname{sign}}^{\varepsilon}_P(x)$ (which does not depend on any choice of labelling for the signature maps).

Proposition 5.4. *Let (A,σ) be an algebra with involution over K. Then for any $x \in \widetilde{W}(A,\sigma)$, the total signature $\widetilde{\operatorname{sign}}(x)$ is a continuous function.*

Proof. By definition, $\widetilde{\operatorname{sign}}(x)^{-1}(\{n\})$ is the intersection of $\widetilde{X}(A,\sigma)$ with the Zariski-closed set $V(x - n\langle 1\rangle)$ in $\operatorname{Spec}(\widetilde{W}(A,\sigma))$, so it is closed in $\widetilde{X}(A,\sigma)$. □

Corollary 5.5. *The topological space $\widetilde{X}(A,\sigma)$ is compact and totally disconnected.*

Proof. Let us consider the function

$$\begin{array}{rccc} F: & \widetilde{X}(A,\sigma) & \longrightarrow & \{-1,0,1\}^{\widetilde{W}(A,\sigma)} \\ & \widetilde{I}_P^{\varepsilon} & \longmapsto & (x \mapsto \tau(\widetilde{\mathrm{sign}}_P^{\varepsilon}(x))) \end{array}$$

where $\tau : \mathbb{Z} \to \{-1,0,1\}$ maps non-zero integers to their sign. Then we claim that F is injective and continuous; since the target space is compact this is a homeomorphism onto its image, which concludes.

The injectivity is clear: if two ideals have the same image, then considering the restriction to elements $\langle a \rangle \in W(K)$ we see that they must correspond to the same $P \in X(K)$, and if the signature maps have the same signs on hermitian forms they are equal.

For continuity, note that F corresponds to the same map

$$\Phi : \widetilde{X}(A,\sigma) \times \widetilde{W}(A,\sigma) \to \{-1,0,1\}$$

as $\tau \circ \widetilde{\mathrm{sign}}$. Since each $\widetilde{\mathrm{sign}}(x)$ is continuous, Φ is continuous for the product topology if we put the discrete topology on $\widetilde{W}(A,\sigma)$, and so F is continuous. □

6 Polarizations

One on the main goals in [2] and [3] can be interpreted as the definition of an appropriate total signature that is defined on $X(K)$ instead of $\widetilde{X}(A,\sigma)$ (this is what they call $\mathcal{M}$-signatures and H-signatures).

Definition 6.1. *Let (A,σ) be an algebra with involution over K. If U is an open subset of $X(K)$, a local polarization of (A,σ) over U is a set-theoretical section of $\overline{\pi}$ on U. We write $\mathrm{Pol}_U(A,\sigma)$ for the set of local polarizations over U. If $s \in \mathrm{Pol}_U(A,\sigma)$, we say that $-s \in \mathrm{Pol}_U(A,\sigma)$, such that $-s(P) \neq s(P)$ for all $P \in U$, is the opposite (local) polarization of s.*

When $U = X(K)$ (resp. $X_+(A)$, $X_-(A)$), we speak of a global (resp. orthogonal, symplectic) polarization of (A,σ), and the set of those is denoted by $\mathrm{Pol}(A,\sigma)$ (resp. $\mathrm{Pol}_+(A,\sigma)$, $\mathrm{Pol}_-(A,\sigma)$). A global polarization is also simply called a polarization.

If $s \in \mathrm{Pol}(A,\sigma)$, then for any $x \in \widetilde{W}(A,\sigma)$, the total signature of x relative to s is

$$\widetilde{\mathrm{sign}}^s(x) : X(K) \xrightarrow{s} \widetilde{X}(A,\sigma) \xrightarrow{\widetilde{\mathrm{sign}}(x)} \mathbb{Z}.$$

We also write $\widetilde{\mathrm{sign}}_P^s(x) = \widetilde{\mathrm{sign}}^s(x)(P)$.

Clearly $\mathrm{Pol}(A,\sigma) \simeq \mathrm{Pol}_+(A,\sigma) \times \mathrm{Pol}_-(A,\sigma)$. The way we see things is that a polarization is the choice of a labelling of $\widetilde{\mathrm{sign}}_P^+$ and $\widetilde{\mathrm{sign}}_P^-$, and an orthogonal (resp. symplectic) polarization is such a choice for only the $P \in X_+(A)$ (resp. $X_-(A)$). The way we defined the signature maps shows that a choice of polarization is also equivalent to a choice of Morita equivalence between (A_{K_P}, σ_{K_P}) and (D_P, θ_P) for all $P \in X(K)$, but the global structure of $\widetilde{X}(A,\sigma)$ makes it much more convenient to discuss polarizations. Our goal is to find relevant natural classes of polarizations, or even ideally natural polarizations on various (A,σ).

Remark 6.2. The notion of $\mathcal{M}$-signature in [2] corresponds to an arbitrary (orthogonal/symplectic) polarization.

Remark 6.3. For any polarization s, and any $x \in W(K)$, $\widetilde{\mathrm{sign}}^s(x)$ is the classical total signature $\mathrm{sign}(x) : X(K) \to \mathbb{Z}$.

There are natural symmetries of polarizations that we want to emphasize. Let G be the group of set-theoretical automorphisms of $\overline{\pi}$, and G_c (for *continuous*) the group of topological automorphisms of $\overline{\pi}$. Then G can be identified with the multiplicative group $\mathcal{F}(X(K), \{\pm 1\})$: a function $f : X(K) \to \{\pm 1\}$ acts by swapping the elements of the fiber above $P \in X(K)$ iff $f(P) = -1$. This is also naturally isomorphic to the group $(\mathcal{P}(X(K)), \Delta)$ of subsets of $X(K)$ with the symmetric difference, is we associate f to $f^{-1}(\{-1\})$. Then $G_c \subset G$ corresponds to the continuous functions inside $\mathcal{F}(X(K), \{\pm 1\})$, and to the clopen subsets inside $\mathcal{P}(X(K))$. Clearly G acts, simply transitively, on $\mathrm{Pol}(A, \sigma)$. If $f \in \mathcal{F}(X(K), \{\pm 1\})$, then its action (as an element of G) on the total signatures is $\widetilde{\mathrm{sign}}^{f\cdot s} = f\widetilde{\mathrm{sign}}^s$. The function $s \mapsto -s$ corresponds to the constant function -1 in $\mathcal{F}(X(K), \{\pm 1\})$, and to $X(K) \in \mathcal{P}(X(K))$.

Note that since $\overline{\pi}$ is the application of the functor Spec_0 to the inclusion $W(K) \to \widetilde{W}(A, \sigma)$, there is a canonical embedding of the $W(K)$-algebra automorphisms of $\widetilde{W}(A, \sigma)$ in G_c. We call G_a (for *algebraic*) the image of the embedding. The image of the subgroup of standard automorphisms is denoted G_s (for *standard*). This action can also be deduced from the fact that by construction, $(A, \sigma) \mapsto \widetilde{X}(A, \sigma)$ defines a functor from $\mathbf{Br}_h(K)$ to the category of sets above $X(K)$. Note that G_s is naturally a quotient of K^* since standard automorphisms have the form $(\langle a \rangle_\sigma)_*$ for some $a \in K^*$.

We then have

$$G_s \subset G_a \subset G_c \subset G,$$

and if we cannot find a canonical element of $\mathrm{Pol}(A, \sigma)$ for an arbitrary (A, σ) we can at least try to find canonical classes in $\mathrm{Pol}(A, \sigma)/H$ for those various subgroups $H \subset G$ (of course $\mathrm{Pol}(A, \sigma)/G = \{*\}$), or maybe at least in $\mathrm{Pol}_\varepsilon(A, \sigma)/H$ for some ε.

Remark 6.4. Note that by functoriality with respect to $\mathbf{Br}_h(K)$, $\mathrm{Pol}(A, \sigma)/G_s$ only depends on the Brauer class $[A]$.

The topological nature of our spaces makes it very natural to investigate the following class:

Definition 6.5. *We write* $\mathrm{Pol}^c(A, \sigma)$ *for the set of continuous polarizations on* (A, σ)*, that is continuous sections of* $\overline{\pi}$*. Likewise, we have* $\mathrm{Pol}^c_+(A, \sigma)$ *and* $\mathrm{Pol}^c_-(A, \sigma)$*, so that* $\mathrm{Pol}^c(A, \sigma) \simeq \mathrm{Pol}^c_+(A, \sigma) \times \mathrm{Pol}^c_-(A, \sigma)$.

Remark 6.6. By construction, a polarization is the same as a set-theoretic section of $\pi : \mathrm{Spec}(\widetilde{W}(A, \sigma)) \to \mathrm{Spec}(W(K))$ that is compatible with the specialization of points. Then a continuous polarization is the same as a continuous section of π.

Proposition 6.7. *Let* (A, σ) *be an algebra with involution over* K*. A polarization* $s \in \mathrm{Pol}(A, \sigma)$ *is continuous iff for all* $x \in \widetilde{W}(A, \sigma)$*, the total signature* $\widetilde{\mathrm{sign}}^s(x)$ *relative to* s *is continuous on* $X(K)$*.*

Proof. Since the absolute total signature $\widetilde{\mathrm{sign}}(x)$ is continuous on $\widetilde{X}(A,\sigma)$ (proposition 5.4), clearly if s is a continuous section of $\overline{\pi}$ then the composition $\widetilde{\mathrm{sign}}^s(x)$ is also continuous.

Conversely, assume all $\widetilde{\mathrm{sign}}^s(x)$ are continuous on $\widetilde{X}(A,\sigma)$. Let $D(x) \subset \mathrm{Spec}(\widetilde{W}(A,\sigma))$ be the open subset defined by x (ie the open subscheme defined by the localization at x). By construction, $D_0(x) := D(x) \cap \widetilde{X}(A,\sigma)$ is the subset on which $\widetilde{\mathrm{sign}}(x)$ takes non-zero values, so $s^{-1}(D_0(x))$ is the subset of $X(K)$ on which $\widetilde{\mathrm{sign}}^s(x)$ takes non-zero values. By hypothesis, it is open in $X(K)$. Since the $D_0(x)$ form an open basis of $\widetilde{X}(A,\sigma)$, this means that s is continuous. □

It follows from the definition of G_c that if $\mathrm{Pol}^c(A,\sigma)$ is not empty, then it is a simply transitive G_c-set, so it defines a class in $\mathrm{Pol}(A,\sigma)/G_c$. Thus we just need to know whether there is one continuous polarization to find all of them. This is strongly related to the study of H-signatures in [3], as we will now investigate.

Definition 6.8. *Let (A,σ) be an algebra with involution over K. If $x \in \widetilde{W}(A,\sigma)$, we write*

$$U(x) = \{P \in X(K) \,|\, \widetilde{\mathrm{sign}}^+_P(x) \neq \widetilde{\mathrm{sign}}^-_P(x)\}.$$

We call $U(x)$ the principal subset of $X(K)$ defined by x.

We also define $s_x \in \mathrm{Pol}_{U(x)}(A,\sigma)$, called the principal local polarization defined by x, as the unique local polarization such that $\widetilde{\mathrm{sign}}^{s_x}_P(x) > \widetilde{\mathrm{sign}}^{-s_x}_P(x)$ for all $P \in U(x)$.

Proposition 6.9. *Let (A,σ) be an algebra with involution over K Then for any $x \in \widetilde{W}(A,\sigma)$, $U(x)$ is a clopen subset of $X(K)$, and $s_x : U(x) \to \widetilde{X}(A,\sigma)$ is a continuous local polarization of (A,σ) over $U(x)$.*

Proof. Let $\tau : \widetilde{X}(A,\sigma) \to \widetilde{X}(A,\sigma)$ be the function that swaps the elements of every fiber of π. It is continuous, for instance because it is induced by the standard automorphism defined by $\langle -1\rangle_\sigma$. We define $f : \widetilde{X}(A,\sigma) \to \mathbb{Z}^2$ by $f = (\widetilde{\mathrm{sign}}(x), \widetilde{\mathrm{sign}}(x) \circ \tau)$, and

$$S = \{(m,n) \in \mathbb{Z}^2 \,|\, m \neq n\}, \quad S^+ = \{(m,n) \in \mathbb{Z}^2 \,|\, m > n\}.$$

Then $U(x) = \pi(f^{-1}(S))$ and $\mathrm{Im}(s_x) = f^{-1}(S^+)$, so $\mathrm{Im}(s_x)$ is closed in $\widetilde{X}(A,\sigma)$ and $U(x)$ is clopen in $X(K)$ (here we use the compacity of $\widetilde{X}(A,\sigma)$). Now if Y is any closed set in $\widetilde{X}(A,\sigma)$, then $s_x^{-1}(Y) = \pi(Y \cap \mathrm{Im}(s_x))$, so it is closed in $X(K)$, which shows that s_x is continuous. □

Then we interpret the results in [3] as:

Theorem 6.10. *Let (A,σ) be an algebra with involution over K. Then for any $x_1,\dots,x_n \in W_\varepsilon(A)$, there exists $x \in W_\varepsilon(A,\sigma)$ such that $U(x_1) \cup \cdots \cup U(x_n) = U(x)$. In particular, there exists $x \in W_\varepsilon(A,\sigma)$ such that $U(x) = X_\varepsilon(A)$, so there exist global continuous polarizations, and $\widetilde{X}(A,\sigma) \approx X(K) \coprod X(K)$ as topological spaces, with $\overline{\pi}$ being the canonical projection.*

Furthermore, the class of global principal polarizations is a transitive G_c-set, thus it is exactly $\mathrm{Pol}^c(A,\sigma)$.

Proof. The existence of $x \in W_\varepsilon(A,\sigma)$ such that $\bigcup U(x_i) = U(x)$ is a reformulation of [3, 3.1]. The existence of $x \in W_\varepsilon(A,\sigma)$ such that $U(x) = X_\varepsilon(A)$ follows by compacity, since lemma 4.4 shows that the $U(x_i)$ form an open cover of $X_\varepsilon(A)$. Since $\mathrm{Pol}^c_+(A,\sigma)$ and $\mathrm{Pol}^c_-(A,\sigma)$ are non-empty, so is $\mathrm{Pol}^c(A,\sigma)$. If $s \in \mathrm{Pol}(A,\sigma)$, then $\widetilde{X}(A,\sigma) = \mathrm{Im}(s) \coprod \mathrm{Im}(-s)$, and if s is continuous, $\mathrm{Im}(s)$ and $\mathrm{Im}(-s)$ are homeomorphic to $X(K)$.

The fact that the principal polarizations are a transitive G_c-set is a reformulation of [3, 3.3]. Since they are included in $\mathrm{Pol}^c(A,\sigma)$ and $\mathrm{Pol}^c(A,\sigma)$ is also a transitive G_c-set, we can conclude. □

Remark 6.11. With this framework, the weaker lemma 4.4 simply states that the $U(x)$ with $x \in W_\varepsilon(A,\sigma)$ form a open cover of $X_\varepsilon(A)$, whichs shows that $\widetilde{X}(A,\sigma)$ is a double cover of $X(K)$, since it has local trivialization. The notion of H-signatures in [2] corresponds to taking a finite open cover of $X_\varepsilon(A)$ by principal subsets (which exists by compacity). In general, H-signatures correspond to continuous polarizations.

Given that $\mathrm{Spec}(\widetilde{W}(A,\sigma))$ is not only a topological space but a scheme, we also have another natural class of polarizations: if $\rho : \widetilde{W}(A,\sigma) \to W(K)$ is a retraction (see definition 3.1), then applying the Spec functor gives a scheme morphism $\rho^* : \mathrm{Spec}(W(K)) \to \mathrm{Spec}(\widetilde{W}(A,\sigma))$, so in particular a continuous polarization. We call polarizations of this form *algebraic polarizations* of (A,σ), and we write $\mathrm{Pol}^a(A,\sigma)$. Similarly, orthogonal (resp. symplectic) retractions define the set $\mathrm{Pol}^a_+(A,\sigma)$ (resp. $\mathrm{Pol}^a_-(A,\sigma)$) of algebraic orthogonal (resp. symplectic) polarizations of (A,σ). There is an obvious injection $\mathrm{Pol}^a(A,\sigma) \subset \mathrm{Pol}^a_+(A,\sigma) \times \mathrm{Pol}^a_-(A,\sigma)$, but it is not clear if it is surjective in general. The existence of an algebraic (global, orthogonal or symplectic) polarization of (A,σ) obviously depends only on the Brauer class $[A]$.

Remark 6.12. By construction, G_a acts on $\mathrm{Pol}^a_+(A,\sigma)$ and $\mathrm{Pol}^a_-(A,\sigma)$, and it acts transitively on $\mathrm{Pol}^a(A,\sigma)$. In particular, there is a well-defined "algebraic" element in $\mathrm{Pol}(A,\sigma)/G_a$, and the sets $\mathrm{Pol}^a(A,\sigma)/G_s$, $\mathrm{Pol}^a_+(A,\sigma)/G_s$ and $\mathrm{Pol}^a_-(A,\sigma)/G_s$ are well-defined, and they only depend on the Brauer class $[A]$.

Example 6.13. If A is split, there are algebraic polarizations of (A,σ), by example 3.2.

Example 6.14. According to proposition 3.3 and remark 3.4, there is always a canonical algebraic symplectic polarization of $(\mathbf{H}_K,\gamma)$, and if K is Pythagorean there is a canonical global algebraic polarization. On the other hand, $(\mathbf{H}_K,\gamma)$ does not have algebraic polarizations if the Pythagoras number of K is at least 3.

We do not know of any other cases where algebraic polarizations exist, and it would be interesting to characterize the Brauer classes for which it is the case.

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