Witt invariants of quaternionic forms

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Introduction

In the seminal [2], Serre starts the study of cohomological invariants, in particuliar of algebraic groups. Let k be a base field, which in this article we will assume to be of characteristic not 2. Then if $\mathbf{Field}_{/k}$ is the category of field extensions of k (it is enough to consider finitely generated extensions, in case one wants to work in an essentially small category), and $F : \mathbf{Field}_{/k} \to \mathbf{Set}$ and $A : \mathbf{Field}_{/k} \to \mathbf{Ab}$ are functors, then an invariant of F with values in A is simply a natural transformation from F to A (seen as a functor to $\mathbf{Set}$). We write $\mathrm{Inv}(F, A)$ for the set of such invariants; this is clearly an abelian group for the pointwise addition coming from the abelian group structure of $A(K)$ for any extension K/k.

If $A(K) = H^i(K, M)$ for some Galois module M defined over k, we speak of cohomological invariants with values in M. If $F(K) = H^1(K, G)$ for some algebraic group G defined over k, we speak of "invariants of G" (a more explicit terminology would be "invariants of G-torsors"). If G is a classical group, one recovers some familiar functors; in particular, if $G = O_n$ is the classical orthogonal group, the corresponding F is isomorphic to the functor $\mathbf{Quad}_n$ of non-singular n-dimensional quadratic forms. Taking $M = \mathbb{Z}/2\mathbb{Z}$ with trivial Galois action, the classical invariants of quadratic forms q such as the determinant $\det(q) \in K^\times/(K^\times)^2 \simeq H^1(K, \mathbb{Z}/2\mathbb{Z})$ and the Hasse invariant in $\mathrm{Br}(K)[2] \simeq H^2(K, \mathbb{Z}/2\mathbb{Z})$ give examples of cohomological invariants of O_n. In [2] Serre gives a complete description of mod 2 cohomological invariants of O_n, proving that they form a free module over $H^*(k, \mathbb{Z}/2\mathbb{Z})$ with basis given by the Stiefel-Whitney invariants w_i (the determinant and Hasse invariant are then recovered as, respectively, w_1 and w_2).

Serre also introduces in [2] the notion of Witt invariants (which amounts to taking $A(K) = W(K)$ the Witt group of K), noticing that the formal properties of the Witt group regarding residues and specialization with respect to discrete valuations are similar enough to cohomology groups that a similar theory can be developped. For Witt invariants of O_n, he proves that they form a free $W(k)$-module generated by the λ-operations λ^i (for $0 \leqslant i \leqslant n$).

Although there is clearly much more literature devoted to cohomological invariants, in particular due to their use for rationality problems and computations of essential dimensions of algebraic groups, the solution of the Milnor Conjecture by Voevodsky gives a direct connexion between cohomological and Witt invariants which can motivate a closer interest in those. Explicitly, if we can define a Witt invariant $\alpha \in \mathrm{Inv}(F, W)$ with values in $I^n \subset W$ (where I^n is the nth power of the fundamental ideal of the Witt group), then composing with the canonical isomorphism $I^n(K)/I^{n+1}(K) \xrightarrow{\sim} H^2(K, \mathbb{Z}/2\mathbb{Z})$ defines a cohomological invariant in $\mathrm{Inv}^n(F, \mathbb{Z}/2\mathbb{Z})$. It is not difficult to realize that one recovers all cohomological invariants of O_n in this way (see [3, Section 9] for instance)

Now we can ask the question of invariants of $G = O(A, \sigma)$ where (A, σ) is a central simple algebra with orthogonal involution. This algebraic group is a form of O_n (which corresponds to the case where $A = M_n(k)$ and σ is the adjoint involution of the form $\langle 1, \dots, 1\rangle$), so we might expect a similar description of invariants. The pointed set $H^1(K, G)$ is this time is a canonical bijection with the set of isometry classes of nondegenerate 1-dimensional hermitian forms over (A, σ) (the base point being $\langle 1\rangle_\sigma$). We address in this article the question of Witt invariants of $O(A, \sigma)$ when A has index $\leqslant 2$ and degree $2r$ (which is a class stable by scalar extension). If Q is the quaternion algebra Brauer-equivalent to A, we can choose a Morita equivalence between (A, σ) and (Q, γ) where γ is the canonical symplectic involution of Q, and this induces an isomorphism between the functor $H^1(-, G)$ and $H_Q^{(r)}$, where for any extension K/k, $H_Q^{(r)}(K)$ is the set of isometry classes of nondegenerate anti-hermitian forms of reduced dimension $2r$ over (Q_K, γ_K). Our main result is then Theorem 4.7, which states that the Witt invariants of $H_Q^{(r)}(K)$ (and therefore of G) are again generated by λ-operations (this time in the sense of [4]), but the coefficients have to be taken not only in $W(k)$ but in the mixed Witt ring $\widetilde{W}^{-1}(Q, \gamma)$ introduced in [5]. Furthermore, such a decomposition is not exactly unique, the norm form n_Q being the obstruction.

Notations and conventions

If Q is a quaternion algebra, γ is its canonical symplectic involution, Q_0 is the space of pure quaternions, and $Q_0^\times$ the set of invertible pure quaternions.

1 Mixed Witt rings

1.1 General case

Let (A, σ) be an Azumaya algebra with involution of the first kind over k, and $\varepsilon \in \mu_2(k)$. Then $SW^\varepsilon(A, \sigma)$ is defined as the monoid of ε-hermitian forms over (A, σ), with orthogonal sums, $GW^\varepsilon(A, \sigma)$ is the Grothendieck group of $SW^\varepsilon(A, \sigma)$, and $W^\varepsilon(A, \sigma)$ is the quotient of $GW^\varepsilon(A, \sigma)$ by the subgroup of hermitian forms.

For any $a \in A^\times$ such that $\sigma(a) = \varepsilon a$, the elementary diagonal form $\langle a\rangle_\sigma$ is

$$\begin{array}{rccc} \langle a\rangle_\sigma : & A \times A & \longrightarrow & A \\ & (x, y) & \longmapsto & \sigma(x)ay. \end{array} \tag{1}$$

The diagonal form $\langle a_1, \dots, a_r\rangle_\sigma$ is the orthogonal sum

$$\langle a_1, \dots, a_r\rangle_\sigma = \langle a_1\rangle_\sigma \perp \cdots \perp \langle a_r\rangle_\sigma. \tag{2}$$

When $(A, \sigma) = (k, \mathrm{Id})$ and $\varepsilon = 1$ we just write $SW(k)$, $GW(k)$ and $W(k)$. Note that $W^\varepsilon(A, \sigma)$ is naturally a $W(k)$-module. We define a $\mathbb{Z}/2\mathbb{Z}$-graded $GW(k)$-module

$$\widetilde{GW}^\varepsilon(A, \sigma) = GW(k) \oplus GW^\varepsilon(A, \sigma) \tag{3}$$

and a $\mathbb{Z}/2\mathbb{Z}$-graded $W(k)$-module

$$\widetilde{W}^{\varepsilon}(A,\sigma) = W(k) \oplus W^{\varepsilon}(A,\sigma). \tag{4}$$

If (A,σ) and (B,τ) are Azumaya algebra with involution of the first kind over k, and $\varepsilon_0 \in \mu_2(k)$, we write

$$(B,\tau) \overset{(V,h)}{\rightsquigarrow} (A,\sigma) \tag{5}$$

if V is a B-A-bimodule, balanced over k, such that $B \simeq \mathrm{End}_A(V)$ for this action, and h is an ε_0-hermitian form over (A,σ), and τ is the adjoint involution of h. We then say that (5) is an ε_0-hermitian Morita equivalence, or simply a Morita equivalence. Such an equivalence induces graded isomorphisms h_* which fit in this commutative diagram

$$\begin{array}{ccc} \widetilde{GW}^{\varepsilon}(B,\tau) & \xrightarrow{h_*} & \widetilde{GW}^{\varepsilon\varepsilon_0}(A,\sigma) \\ \downarrow & & \downarrow \\ \widetilde{W}^{\varepsilon}(B,\tau) & \xrightarrow{h_*} & \widetilde{W}^{\varepsilon\varepsilon_0}(A,\sigma) \end{array} \tag{6}$$

such that h_* is the identity on the even components $GW(k)$ and $W(k)$.

In [5], a graded commutative ring structure is defined on $\widetilde{GW}^{\varepsilon}(A,\sigma)$ and $\widetilde{W}^{\varepsilon}(A,\sigma)$ such that (6) is a commutative diagram of rings. The product is characterized by

$$\langle a\rangle_\sigma \cdot \langle b\rangle_\sigma \simeq T_{\sigma,a,b} \in SW(k) \tag{7}$$

where $T_{\sigma,a,b}$ is the twisted involution trace form defined as

$$\begin{array}{rccl} T_{\sigma,a,b}: & A\times A & \longrightarrow & k \\ & (x,y) & \longmapsto & \mathrm{Trd}_A(\sigma(x)ay\sigma(b)). \end{array} \tag{8}$$

Furthermore, if K/k is a field extension, then the scalar extension maps induce a commutative diagram

$$\begin{array}{ccc} \widetilde{GW}^{\varepsilon}(B,\tau) & \xrightarrow{h_*} & \widetilde{GW}^{\varepsilon\varepsilon_0}(A,\sigma) \\ \downarrow & & \downarrow \\ \widetilde{GW}^{\varepsilon}(B_K,\tau_K) & \xrightarrow{(h_K)_*} & \widetilde{GW}^{\varepsilon\varepsilon_0}(A_K,\sigma_K) \end{array} \tag{9}$$

and similarly for mixed Witt rings.

If $(A,\sigma) = (k,\mathrm{Id})$ and $\varepsilon = 1$, the canonical $\mathbb{Z}/2\mathbb{Z}$-graded $W(k)$-module isomorphism

$$\widetilde{W}^1(k,\mathrm{Id}) = W(k)\oplus W(k) \simeq W(k)[\mathbb{Z}/2\mathbb{Z}] \tag{10}$$

is a graded $W(k)$-algebra isomorphism. Let us write

$$\delta : \widetilde{W}^1(k,\mathrm{Id}) \to W(k) \tag{11}$$

for the map $W(k)\oplus W(k) \to W(k)$ given by the sum of components. It is a $W(k)$-algebra morphism.

1.2 Quaternion algebras

We consider the case of a quaternion algebra Q with its canonical involution γ, and $\varepsilon = -1$. Then for any invertible pure quaternions $z_1, z_2 \in Q_0^\times$, a direct computation of the form T_{γ,z_1,z_2} in (8) (see [5, Prop 4.12]) shows that we have in $\widetilde{W}^{-1}(Q,\gamma)$:

$$\langle z_1\rangle_\gamma \cdot \langle z_2\rangle_\gamma = \langle -\operatorname{Trd}_Q(z_1z_2)\rangle \varphi_{z_1,z_2} \in W(k) \tag{12}$$

where φ_{z_1,z_2} is the unique 2-fold Pfister form whose Witt class is $\langle\langle z_1^2, z_2^2\rangle\rangle - n_Q \in W(k)$. If $\operatorname{Trd}_Q(z_1z_2) = 0$ (which means that z_1 and z_2 anti-commute), (12) should be understood as saying that $\langle z_1\rangle_\gamma \cdot \langle z_2\rangle_\gamma = 0 \in W(k)$.

When Q is not split, (12) entirely characterizes the $W(k)$-algebra structure of $\widetilde{W}^{-1}(Q,\gamma)$ since $W^{-1}(Q,\gamma)$ is additively generated by the $\langle z\rangle_\gamma$. When Q is split, we may choose $z_0 \in Q \setminus \{0\}$ such that $z_0^2 = 0$. Then the left ideal Qz_0 is a 2-dimensional k-vector space, and if we define the anti-symmetric bilinear form

$$\begin{array}{rccc} b_{z_0}: & Qz_0 \times Qz_0 & \longrightarrow & k \\ & (zz_0, z'z_0) & \longmapsto & -z_0\gamma(z)z', \end{array} \tag{13}$$

or equivalently

$$b(x,y)z_0 = \gamma(x)y \tag{14}$$

for all $x, y \in Qz_0$, we get an anti-hermitian Morita equivalence

$$(Q,\gamma) \overset{(Qz_0,b_{z_0})}{\rightsquigarrow} (k,\mathrm{Id}). \tag{15}$$

This induces a $W(k)$-algebra morphism

$$\Phi_{z_0} : \widetilde{W}^{-1}(Q,\gamma) \overset{(b_{z_0})_*}{\overset{\sim}{\to}} \widetilde{W}^1(k,\mathrm{Id}) \overset{\sim}{\to} W(k)[\mathbb{Z}/2\mathbb{Z}] \overset{\delta}{\to} W(k) \tag{16}$$

using (6), (10) and (11). Note that the restriction of Φ_{z_0} to $W(k)$ is the identity.

Lemma 1.1. *Let $z_0 \in Q_0 \setminus \{0\}$ be such that $z_0^2 = 0$. Then for any $z \in Q_0^\times$, $(b_{z_0})_*(\langle z\rangle_\gamma)$ is isometric to the symmetric bilinear form*

$$\begin{array}{rccc} b_{z_0,z}: & Qz_0 \times Qz_0 & \longrightarrow & k \\ & (z_1z_0, z_2z_0) & \longmapsto & -\operatorname{Trd}_Q(z_0\gamma(z_1)zz_2). \end{array} \tag{17}$$

If z and z' anti-commute, this form is a hyperbolic plan; otherwise, it is isometric to $\langle -\operatorname{Trd}_Q(zz_0)\rangle\langle\langle z^2\rangle\rangle$.

Proof. From the general theory of hermitian Morita equivalences, $(b_{z_0})_*$ sends the anti-hermitian space $(Q, \langle z\rangle_\gamma)$ to

$$\begin{array}{ccc} Q\otimes_Q Qz_0 \times Q\otimes_Q Qz_0 & \longrightarrow & K \\ (z_1\otimes z_0, z_2\otimes z_0) & \longmapsto & b_{z_0}(z_0, \langle z\rangle_\gamma(z_1,z_2)z_0). \end{array}$$

Now

$$b_{z_0}(z_0, \langle z\rangle_\gamma(z_1,z_2)z_0)z_0 = -z_0(\gamma(z_1)zz_2)z_0 \qquad = -\operatorname{Trd}_Q(z_0\gamma(z_1)zz_2)z_0$$

where the last equality is because for any $x \in Q$ we have

$$\operatorname{Trd}_Q(z_0x)z_0 = (z_0x - \gamma(x)z_0)z_0 = z_0xz_0.$$

If z and z_0 anti-commute, $zz_0 \neq 0$ and $b_{z_0,z}(zz_0, zz_0) = 0$, so $b_{z_0,z}$ is isotropic, and therefore a hyperbolic plan.

If z and z_0 do not anti-commute, we have $\mathrm{Trd}_Q(zz_0) \neq 0$, and (z_0, zz_0) is an orthogonal k-basis of Qz_0 for $b_{z_0,z}$, which gives the diagonalization $b_{z_0,z} \simeq \langle -\mathrm{Trd}_Q(zz_0), \mathrm{Trd}_Q(zz_0)z^2\rangle$. □

1.3 λ-operations

In [4], for any Azumaya algebra with involution of the first kind (A,σ) over k, and any $\varepsilon \in \mu_2(k)$, a structure of pre-λ-ring (see [9] for a reference about pre-λ-rings) is defined on $\widetilde{GW}^{\varepsilon}(A,\sigma)$, whose restriction to $GW(K)$ is the usual λ-ring structure (studied for instance in [7]).

It is compatible with Morita equivalences, meaning that the top row of (6) is an isomorphism of pre-λ-rings, and it is compatible with scalar extensions, meaning the (9) is a commutative diagram of pre-λ-rings.

Note that the pre-λ-ring structure is compatible with the $\mathbb{Z}/2\mathbb{Z}$-grading, meaning that $\lambda^d(GW^{\varepsilon}(A,\sigma))$ is included in $GW(k)$ when d is even, and in $GW^{\varepsilon}(A,\sigma)$ when d is odd. Also note that by definition of a pre-λ-ring, λ^0 is the constant function to $\langle 1\rangle$, and λ^1 is the identity.

It follows from [4, Prop 5.2] that if σ is symplectic, $a \in A^{\times}$ is ε-symmetric and $n = \deg(A)$, then

$$\lambda^n(\langle a\rangle_\sigma) = \langle \mathrm{Nrd}_A(a)\rangle. \tag{18}$$

The square class defined by this 1-dimensional form is precisely the determinant of $\langle a\rangle_\sigma$ (as defined in [6]).

2 Generic splitting and residues

A crucial method for us is the scalar extension to a generic splitting field of our quaternion algebra. The behaviour of anti-hermitian forms under such generic splitting has been the object of a fair amount of research, but we will mainly refer to [8], which presents a good overview of the situation.

2.1 The generic elementary form

Let Q be a quaternion algebra. We choose a quaternionic basis (i,j,ij), with $i^2 = a$ and $j^2 = b$, such that $(ij)^2$ is not a square in k. This is of course automatic when Q is not split, and even when Q is split it is always possible unless k is quadratically closed (we exclude this case from the present discussion).

We define the generic pure quaternion of Q as

$$\widetilde{\omega} = xi + yj + zij \in Q_{k(x,y,z)}. \tag{19}$$

To make use of the fact that $\widetilde{\omega}$ is generic, we use the setting of versal torsors as in [2, Section 5]. Let $h_0 \in H^{(1)}_Q(k)$ and $G = O(h_0)$ be its orthogonal group. There is a canonical isomorphism $h \mapsto \mathrm{Iso}(h,h_0)$ between the functors $H^{(1)}_Q$ and $K \mapsto H^1(K,G)$ which allows us to view elementary forms as G-torsors. Then $\langle\widetilde{\omega}\rangle_{\gamma_{k(x,y,z)}} \in H^{(1)}_Q(k(x,y,z))$ is a torsor over the function field of $\mathbb{A}^3_k$, and it is the generic point of a torsor over $\mathbb{A}^3_k \setminus V(\widetilde{\omega}^2)$ (because the specialization of $\widetilde{\omega}$ at some point in $\mathbb{A}^3_k$ is non-invertible if and only if this point is in $V(\widetilde{\omega}^2)$). Note that $\widetilde{\omega}^2 \in k[x,y,z]$ is nothing but the pure norm form of Q in the coordinate system given by the basis (i,j,ij) of Q_0.

Lemma 2.1. *The G-torsor over $k(x,y,z)$ corresponding to $\langle\widetilde{\omega}\rangle_{\gamma_{k(x,y,z)}}$ is a versal G-torsor.*

Proof. Let K/k be a field extension with K infinite, and let $h \in H^{(1)}_Q(K)$. Clearly the points $(s,t,r) \in \mathbb{A}^3(K)$ such that $h \simeq \langle si+tj+rij\rangle_{\gamma_K}$ are dense, so for any open $U \subset \mathbb{A}^3_k$ there is a point in $U(K)$ such that h is the corresponding specialization of $\langle\widetilde{\omega}\rangle_{\gamma_{k(x,y,z)}}$ seen as a torsor on $\mathbb{A}^3 \setminus V(\widetilde{\omega}^2)$. □

We also define

$$\omega = xi + yj + ij \in Q_{k(x,y)} \tag{20}$$

and

$$\Delta = -\omega^2 = -ax^2 - by^2 + ab =\in k[x,y]. \tag{21}$$

2.2 The Severi-Brauer variety

Let $SB(Q)$ be the Severi-Brauer variety of Q. By definition, if K/k is an extension, $SB(Q)(K)$ is the set of left ideals of reduced dimension 1 (equivalently, of K-dimension 2) of Q_K. If $I \subset Q_K$ is such an ideal, then $I = Qz_0$ for some non-zero pure quaternion $z_0 \in Q_K$ with $z_0^2 = 0$, and z_0 is unique up to a constant. Thus all such z_0 lie on a line $L(I)$ which is recovered intrinsically as $L(I) = I \cap \gamma_K(I)$.

If X_Q is the projective conic defined by the pure norm form of Q, $X_Q(K)$ is the set of lines in $(Q_K)_0$ consisting of pure quaternion whose square is 0, and there is a canonical isomorphism $SB(Q) \simeq X_Q$ sending $I \in SB(Q)(K)$ to $L(I) \in X_Q(K)$.

Let F_∞ be the quadratic extension $k(ij) \subset Q$ of k. Let $V = ki \oplus kj \subset Q_0$; we define $\mu : V \otimes_k F_\infty \to V$ as the multiplication map inside Q, and $L_\infty = \ker(\mu) \subset Q_0 \otimes_k F_\infty$. Then one may chech that $L\infty$ is a point in $X_Q(F_\infty)$, which defines a closed point $\infty \in X_Q$ of degree 2 and residue field F_∞.

Then if

$$Y = V(\Delta) \subset \mathbb{A}^2_k \tag{22}$$

is the affine conic defined by Δ, there is a natural identification $Y \simeq X_Q \setminus \{\infty\}$, and therefore

$$F = \mathrm{Frac}(k[x,y]/(\Delta)) = k(Y) \tag{23}$$

is a function field of X_Q, and thus a generic splitting field of Q.

The image of $\omega \in Q \otimes_k k[x,y]$ in Q_F is written $\overline{\omega}$. By definition, of F $\overline{\omega}^2 = 0$ (which is a witness to the fact that F is a splitting field of Q).

2.3 Valuations and residues

Let K/k be a field extension and $v : K^\times \to \mathbb{Z}$ a valuation on K which is trivial on k, with valuation ring $\mathcal{O}_v$, uniformizing element π and residue field κ_v. Recall that there are residue maps $\partial^1_v : W(K) \to W(\kappa_v)$ (independent of π) and $\partial^2_{v,\pi} : W(K) \to W(\kappa_v)$ (which depends on the choice of π, but its kernel doesn't). They actually form a $W(k)$-algebra morphism $\partial_{v,\pi} : W(K) \to W(\kappa_v)[\mathbb{Z}/2\mathbb{Z}]$

(where the even component is ∂_v^1 and the odd component is $\partial_{v,\pi}^2$). In practice, if $q \in W(K)$, we can write $q = \langle a_1, \dots, a_n\rangle + \langle \pi\rangle\langle b_1, \dots, b_m\rangle$ with $a_i, b_i \in \mathcal{O}_v$ for all i and j, and then $\partial^1(q) = \langle \overline{a_1}, \dots, \overline{a_n}\rangle$ and $\partial_{2,\pi}(q) = \langle \overline{b_1}, \dots, \overline{b_m}\rangle$.

Every closed point $p \in Y^{(1)}$ defines a discrete rank 1 valuation v_p on F, with residue field F_p. We write

$$W_0(F) = \bigcap_{p \in Y^{(1)}} \ker(\partial^2_{v_p,\pi_p} : W(F) \to W(F_p)) \tag{24}$$

which does not depend on the choice of uniformizers π_p for each $p \in Y^{(1)}$. It is a sub-$W(k)$-algebra of $W(F)$.

There is also the valuation "at infinity" v_∞ corresponding to the point $\infty \in X_Q^{(1)}$, with residue field F_∞. It is characterized by the fact that if $\overline{u} \in k[Y] = k[x,y]/(\Delta)$ is the class of $u = k[x,y]$, then $v_\infty(\overline{u}) = -\deg(u)$. We will shorten $\partial^1_{v_\infty}$ and $\partial^2_{v_\infty,\pi_\infty}$ as ∂^1_∞ and ∂^2_∞, where π_∞ is any choice of uniformizer (which will not matter to us).

2.4 Generic splitting of hermitian forms

Since $\overline{\varpi}^2 = 0$ in Q_F, we get a $W(k)$-algebra morphism

$$\Phi_{\overline{\varpi}} : \widetilde{W}^{-1}(Q_F, \gamma_F) \to W(F) \tag{25}$$

(see (16)), and its composition with the scalar extension map yields a $W(k)$-algebra morphism

$$\Psi_{\overline{\varpi}} : \widetilde{W}^{-1}(Q, \gamma) \to \widetilde{W}^{-1}(Q_F, \gamma_F) \xrightarrow{\Phi_{\overline{\varpi}}} W(F). \tag{26}$$

By definition, the restriction of $\Psi_{\overline{\varpi}}$ to $W(k)$ is the scalar extension map $W(k) \to W(F)$, and its restriction to $W^{-1}(Q,\gamma)$ is the composition of the scalar extension map to F with the isomorphism $(b_{\overline{\varpi}})_*$ (see (13)).

The exact sequences in [8, Thm 5.1, Thm. 5.2] have the following exact sequences as direct consequences:

$$0 \to n_Q W(k) \to W(k) \xrightarrow{\Psi_{\overline{\varpi}}} W_0(F) \xrightarrow{\partial^2_\infty} W(F_\infty) \tag{27}$$

$$0 \to W^{-1}(Q,\gamma) \xrightarrow{\Psi_{\overline{\varpi}}} W_0(F) \xrightarrow{\partial^1_\infty} W(F_\infty) \xrightarrow{s_*} W(k) \tag{28}$$

where $s : F_\infty \to k$ is any k-linear form which is 0 on k. We collect some immediate observations on these sequences:

Proposition 2.2. *We have $W(k) \cap \operatorname{Ker}(\Psi_{\overline{\varpi}}) = n_Q W(k)$ and $W^{-1}(Q,\gamma) \cap \operatorname{Ker}(\Psi_{\overline{\varpi}}) = 0$. The scalar extension map $\widetilde{W}^{-1}(Q,\gamma) \to \widetilde{W}^{-1}(Q_F,\gamma_F)$ has kernel $n_Q W(k)$.*

Corollary 2.3. *We have $n_Q W^{-1}(Q,\gamma) = 0$, and $n_Q \widetilde{W}^{-1}(Q,\gamma) = n_Q W(k)$.*

Proof. Since $\Psi_{\overline{\varpi}}(n_Q) = n_{Q_F} = 0$, $\Psi_{\overline{\varpi}}(n_Q W^{-1}(Q,\gamma)) = 0$, so $n_Q W^{-1}(Q,\gamma)$ since $\Psi_{\overline{\varpi}}$ is injective on $W^{-1}(Q,\gamma)$. □

Remark 2.4. In particular, this means that the $W(k)$-module structure of $W^{-1}(Q,\gamma)$ factors through a $W(k)/(n_Q)$-module structure.

A more precise description of the image and kernel of $\Psi_{\overline{\omega}}$ is:

Proposition 2.5. *The scalar extension map* $\widetilde{W}^{-1}(Q,\gamma) \to \widetilde{W}^{-1}(Q_F,\gamma_F)$ *has kernel* $n_Q W(k)$*, and there is an exact sequence*

$$0 \to (\langle 2\rangle\langle\langle (ij)^2\rangle\rangle - \langle ij\rangle_\gamma)\widetilde{W}^{-1}(Q,\gamma) \to \widetilde{W}^{-1}(Q,\gamma) \xrightarrow{\Psi_{\overline{\omega}}} W_0(F) \to 0.$$

Proof. From the exact sequences (27) and (28), we see that the kernel of $W(k) \to W(F)$ is $n_Q W(k)$, that $W^{-1}(Q,\gamma) \to W^{-1}(Q_F,\gamma_F)$ is injective, and that $\Psi_{\overline{\omega}}$ has image in $W_0(F) \subset W(F)$.

Let us show that $W_0(F)$ is included in the image of $\Psi_{\overline{\omega}}$. Let $q \in W_0(F)$, and write $q_\infty = \partial^1_\infty(q) \in W(F_\infty)$. From (28), we get that $s_*(q_\infty) = 0 \in W(k)$. But since F_∞ is quadratic extension of k, the scalar extension map and s_* fit in an exact sequence

$$W(k) \xrightarrow{\rho} W(F_\infty) \xrightarrow{s_*} W(k)$$

by [1, Thm 34.4]. Thus $q_\infty = \rho(q_0)$ for some $q_0 \in W(k)$. Now let $q_1 = q - \Psi_{\overline{\omega}}(q_0) \in W_0(F)$. Then

$$\partial^1_\infty(q_1) = \partial^1_\infty(q) - \rho(q_0) = 0$$

where we used that the composition

$$W(k) \xrightarrow{\Psi_{\overline{\omega}}} W(F) \xrightarrow{\partial^1_\infty} W(F_\infty)$$

is nothing but ρ by definition of ∂^1_∞. Using (28), we see that $q_1 = \Psi_\omega(h_1)$ for some $h_1 \in W^{-1}(Q,\gamma)$. In the end $q = \Psi_{\overline{\omega}}(q_0 + h_1)$.

Then we prove that $\langle\langle (ij)^2\rangle\rangle - \langle ij\rangle_\gamma \in \mathrm{Ker}(\Psi_{\overline{\omega}})$. By Lemma 1.1, we have

$$\begin{aligned}\Psi_{\overline{\omega}}(\langle ij\rangle_\gamma) &= (b_{\overline{\omega}})_*(\langle ij\rangle_{\gamma_F}) \\ &\simeq \langle -\mathrm{Trd}_{Q_F}(ij\cdot\overline{\omega})\rangle\langle\langle (ij)^2\rangle\rangle \\ &= \langle -2(ij)^2\rangle\langle\langle (ij)^2\rangle\rangle \\ &= \langle 2\rangle\langle\langle (ij)^2\rangle\rangle.\end{aligned}$$

Finally, let $q-h \in \mathrm{Ker}(\Psi_{\overline{\omega}})$. Then $q_F = \Psi_{\overline{\omega}}(h)$ so by (28) we have $\partial^1_\infty(q_F) = 0$, so $q_{F_\infty} = 0$. By [1, Thm 34.7], $q = \langle\langle (ij)^2\rangle\rangle q'$ for some $q' \in W(k)$. Then $\Psi_{\overline{\omega}}(h) = \Psi_{\overline{\omega}}(\langle 2\rangle q'\langle ij\rangle_\gamma)$, so by injectivity of $\Psi_{\overline{\omega}}$ on $W^{-1}(Q,\gamma)$, $h = \langle 2\rangle q'\langle ij\rangle_\gamma$ and $q - h = \langle 2\rangle q'(\langle 2\rangle\langle\langle (ij)^2\rangle\rangle - \langle ij\rangle_\gamma)$. □

3 Modules of invariants

In this section we define various modules of invariants, and prove some general statements that relate them, independently of any choice of generators.

3.1 Definition of the modules

We promote $\widetilde{W}^{-1}(Q,\gamma)$ to a functor $\mathbf{Field}_{/k} \to \mathbf{Ab}$ by setting

$$\underline{W^{-1}(Q,\gamma)} : K/k \mapsto W^{-1}(Q_K,\gamma_K) \tag{29}$$

with the obvious scalar extension morphisms.

Let us then define:

$$I_Q^{(r)} = \mathrm{Inv}\left(H_Q^{(r)}, \underline{\widetilde{W}^{-1}(Q,\gamma)}\right) \tag{30}$$

$$\overline{I}_Q^{(r)} = \mathrm{Inv}\left(H_Q^{(r)}, \underline{\widetilde{W}^{-1}(Q,\gamma)}/(n_Q)\right) \tag{31}$$

$$J_Q^{(r)} = \mathrm{Inv}\left(\left(H_Q^{(1)}\right)^r, \underline{\widetilde{W}^{-1}(Q,\gamma)}\right) \tag{32}$$

$$\overline{J}_Q^{(r)} = \mathrm{Inv}\left(\left(H_Q^{(1)}\right)^r, \underline{\widetilde{W}^{-1}(Q,\gamma)}/(n_Q)\right). \tag{33}$$

These are all $\widetilde{W}^{-1}(Q,\gamma)$-modules. Since the functors where those invariants take values are $\mathbb{Z}/2\mathbb{Z}$-graded, this induces a $\mathbb{Z}/2\mathbb{Z}$-grading on those modules, and we will write

$$I_Q^{(r)} = {}^0I_Q^{(r)} \oplus {}^1I_Q^{(r)} \tag{34}$$

for the corresponding decomposition into even and odd component, and likewise for the other modules. In the end, the module we are truly interested in is ${}^0I_Q^{(r)} = \mathrm{Inv}(H_Q^{(r)}, W)$, but it is necessary to study the full $I_Q^{(r)}$, which is reduced to the study of $\overline{I}_Q^{(r)}$, and in turn of $\overline{J}_Q^{(r)}$, which is determined by induction from $\overline{I}_Q^{(1)}$. Note that by definition $J_Q^{(1)} = I_Q^{(1)}$ and $\overline{J}_Q^{(1)} = \overline{I}_Q^{(1)}$.

Remark 3.1. By Corollary 2.3, we have ${}^1I_Q^{(r)} = {}^1\overline{I}_Q^{(r)}$ and ${}^1J_Q^{(r)} = {}^1\overline{J}_Q^{(r)}$.

There is an obvious surjective natural transformation

$$\begin{array}{ccc} \left(H_Q^{(1)}\right)^r & \longrightarrow & H_Q^{(r)} \\ (h_1,\dots,h_r) & \longmapsto & h_1 \perp \cdots \perp h_r \end{array} \tag{35}$$

and an exact sequence

$$0 \to n_Q W \to \underline{\widetilde{W}^{-1}(Q,\gamma)} \to \underline{\widetilde{W}^{-1}(Q,\gamma)}/(n_Q) \to 0 \tag{36}$$

which together induce a commutative diagram with exact lines and injective vertical arrows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Inv}(H_Q^{(r)}, n_Q W) & \longrightarrow & I_Q^{(r)} & \longrightarrow & \overline{I}_Q^{(r)} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Inv}\left(\left(H_Q^{(1)}\right)^r, n_Q W\right) & \longrightarrow & J_Q^{(r)} & \longrightarrow & \overline{J}_Q^{(r)} \end{array} \tag{37}$$

3.2 Invariants in n_QW

Proposition 3.2. *Every invariant in* $\mathrm{Inv}(H_Q^{(1)}, n_Q W)$ *is constant.*

Proof. Let $\alpha \in \mathrm{Inv}(H_Q^{(1)}, n_Q W)$, which we can see as an invariant in $\mathrm{Inv}(H_Q^{(1)}, W)$. Let $K = k(x,y,z)$. By [2, Cor 27.13], since $\widetilde{h} = \langle \widetilde{\omega} \rangle_{\gamma_K}$ is versal by Lemma 2.1, α is constant if and only if $\alpha(\widetilde{h}) \in W(K)$ is in the image of $W(k) \to W(K)$. This is the case if and only if $\alpha(\widetilde{h})$ is unramified along all hypersurfaces of $\mathbb{A}_k^3$ (see [2, 27.8]). Since $\widetilde{h}$ corresponds to a torsor over $\mathbb{A}_k^3 \setminus V(\widetilde{\omega}^2)$, $\alpha(\widetilde{h})$ can only be ramified along $V(\widetilde{\omega}^2)$ ([2, Thm 27.11]). But by hypothesis, $\alpha(\widetilde{h}) = n_{Q_K} q$ for some $q \in W(K)$. Let $\mathcal{O} = k[x,y,z]_{(\widetilde{\omega}^2)}$ be the valuation ring of the $\widetilde{\omega}^2$-adic valuation of $k[x,y,z]$. We can write $q = q_0 + \langle \widetilde{\omega}^2 \rangle q_1$ with $q_0, q_1 \in W(\mathcal{O})$, and since $\langle \widetilde{\omega}^2 \rangle n_{Q_K} = n_{Q_K}$ because $-\widetilde{\omega}^2$ is represented by n_{Q_K} (it is the reduced norm of $\widetilde{\omega}$), we have $q \in W(\mathcal{O})$, ie it is unramified along $V(\widetilde{\omega}^2)$. □

We present a setting to use induction arguments for invariants. Let $F : \mathbf{Field}_{/k} \to \mathbf{Set}$ and $A : \mathbf{Field}_{/k} \to \mathbf{Ab}$ be functors. We write $\underline{\mathrm{Hom}}$ for the internal Hom in a functor category, and $\underline{\mathrm{Hom}}_{\mathbb{Z}}$ for the internal Hom between two functors with values in abelian groups. By definition, $\mathrm{Inv}_K(F, A) = \underline{\mathrm{Hom}}(F, A)(K)$.

Let X be a finite set and $r \in \mathbb{N}$. There is a canonical map

$$\underline{\mathrm{Hom}}_{\mathbb{Z}}(A^X, \underline{\mathrm{Hom}}(F, A)) \to \underline{\mathrm{Hom}}_{\mathbb{Z}}(A^{X^r}, \underline{\mathrm{Hom}}(F^r, A)) \tag{38}$$

defined through the isomorphisms $\underline{\mathrm{Hom}}_{\mathbb{Z}}(A^X, \underline{\mathrm{Hom}}(F, A)) \simeq \underline{\mathrm{Hom}}(F \times X, \underline{\mathrm{Hom}}_{\mathbb{Z}}(A, A))$ and $\underline{\mathrm{Hom}}_{\mathbb{Z}}(A^{X^r}, \underline{\mathrm{Hom}}(F^r, A)) \simeq \underline{\mathrm{Hom}}(F^r \times X^r, \underline{\mathrm{Hom}}_{\mathbb{Z}}(A, A))$, where X is seen as a constant functor, as well as the composition

$$\underline{\mathrm{Hom}}(F \times X, \underline{\mathrm{Hom}}_{\mathbb{Z}}(A, A)) \to \underline{\mathrm{Hom}}((F \times X)^r, \underline{\mathrm{Hom}}_{\mathbb{Z}}(A, A)^r) \to \underline{\mathrm{Hom}}((F \times X)^r, \underline{\mathrm{Hom}}_{\mathbb{Z}}(A, A)) \tag{39}$$

where the first map is the diagonal embedding, and the second one is induced by the composition map $\underline{\mathrm{Hom}}_{\mathbb{Z}}(A, A)^r \to \underline{\mathrm{Hom}}_{\mathbb{Z}}(A, A)$ (that is, $(f_1, \dots, f_r) \mapsto f_1 \circ \cdots \circ f_r$).

It is an easy fact to prove that under the canonical map (38), an isomorphism $A^X \xrightarrow{\sim} \underline{\mathrm{Hom}}(F, A)$ is sent to an isomorphism $A^{X^r}, \underline{\mathrm{Hom}}(F^r, A)$. There are two special cases that are of interest to us, and we highlight them as lemmas.

Lemma 3.3. *If the canonical map $A \to \underline{\mathrm{Hom}}(F, A)$ is an isomorphism, which means all invariants in $\mathrm{Inv}_K(F, A)$ are constant for all K/k, then for any $r \in \mathbb{N}$, all invariants in $\mathrm{Inv}_K(F^r, A)$ are also constant.*

Proof. It is straightforward to see that (38) (with $X = \{*\}$) sends the canonical map $A \to \underline{\mathrm{Hom}}(F, A)$ to the canonical map $A \to \underline{\mathrm{Hom}}(F^r, A)$, as it corresponds to the constant map $\underline{\mathrm{Hom}}(F \times X, \underline{\mathrm{Hom}}_{\mathbb{Z}}(A, A))$ to the identity of A, and the r-fold composition of the identity is the identity. □

Lemma 3.4. *Suppose A is actually a functor to the category of commutative rings. For any finite family $(\alpha_x)_{x \in X} \in \mathrm{Inv}(F, A)^X$, we define $(\alpha_{\overline{x}})_{\overline{x} \in X^r} \in \mathrm{Inv}(F^r, A)^{X^r}$, where*

$$\alpha_{(x_1, \dots, x_r)}(f_1, \dots, f_r) = \prod_{i=1}^r \alpha_{x_i}(f_i).$$

Assume that $(\alpha_x)_{x \in X}$ is a strong basis of $\mathrm{Inv}(F, A)$, meaning that it is a basis as an $A(k)$-module which remains an $A(K)$-basis of $\mathrm{Inv}_K(F, A)$ for all K/k. Then $(\alpha_{\overline{x}})_{\overline{x} \in X^r}$ is a strong basis of $\mathrm{Inv}(F^r, A)$.

Proof. The family $(\alpha_x)_{x\in X}$ defines a map $A^X \to \underline{\mathrm{Hom}}(F,A)$, and one easily checks that (38) sends it to the map $A^{X^r} \to \underline{\mathrm{Hom}}(F^r,A)$ defined by $(\alpha_{\overline{x}})_{\overline{x}\in X^r}$. Then we may conclude as $(\alpha_x)_{x\in X}$ is a strong basis if and only if the corresponding $A^X \to \underline{\mathrm{Hom}}(F,A)$ is an isomorphism. □

Then we can use our induction properties to prove:

Proposition 3.5. *For any $r\in\mathbb{N}$, all invariants in* $\mathrm{Inv}\left(\left(H_Q^{(1)}\right)^r, n_QW\right)$ *and* $\mathrm{Inv}(H_Q^{(r)}, n_QW)$ *are constant. An invariant in $I_Q^{(r)}$ is constant if and only if its image in $\overline{I}_Q^{(r)}$ is constant.*

Proof. From Proposition 3.2, the hypothesis of Lemma 3.3 is satisfied for $F = H_Q^{(1)}$ and $A = n_QW$, which settles the case of $\mathrm{Inv}\left(\left(H_Q^{(1)}\right)^r, n_QW\right)$. Since $\mathrm{Inv}(H_Q^{(r)}, n_QW)$ is embedded in $\mathrm{Inv}\left(\left(H_Q^{(1)}\right)^r, n_QW\right)$ (see 37), those invariants are also constant.

The second statement is a straightfoward consequence of the first one, using the top exact row in (see 37). □

3.3 Generic splitting

Our description of $\overline{I}_Q^{(1)}$ will rely on generic splitting to reduce to our knowledge of invariants of Quad_2.

Let $\alpha\in\overline{I}_Q^{(r)}$, and let L/F be some extension. We define a function

$$\xi^{(r)}(\alpha) : \mathrm{Quad}_{2r}(L) \to W(L) \tag{40}$$

through the commutative diagram

$$\begin{array}{ccc} H_{Q_F}^{(r)}(L) & \xrightarrow{\alpha} & \widetilde{\mathbf{W}}^{-1}(Q_L,\gamma_L) \\ \big\downarrow{\scriptstyle (b_{\varpi_L})_*} & & \big\downarrow{\scriptstyle \Phi_{\varpi_L}} \\ \mathrm{Quad}_{2r}(L) & \xrightarrow{\xi^{(r)}(\alpha)} & W(L). \end{array} \tag{41}$$

This defines an invariant $\xi^{(r)}(\alpha)\in \mathrm{Inv}_F(\mathrm{Quad}_{2r}, W)$ because of the compatibility of Morita equivalences with scalar extensions, expressed in the commutative diagram (9).

We have thus defined a map

$$\xi^{(r)} : \overline{I}_Q^{(r)} \to \mathrm{Inv}_F(\mathrm{Quad}_{2r}, W). \tag{42}$$

Similarly, we define

$$\zeta^{(r)} : \overline{J}_Q^{(r)} \to \mathrm{Inv}_F(\mathrm{Quad}_2^r, W) \tag{43}$$

through a diagram

$$\begin{array}{ccc} H_{Q_F}^{(1)}(L)^r & \xrightarrow{\alpha} & \widetilde{\mathbf{W}}^{-1}(Q_L,\gamma_L) \\ \big\downarrow{\scriptstyle (b_{\varpi_L})_*^r} & & \big\downarrow{\scriptstyle \Phi_{\varpi_L}} \\ \mathrm{Quad}_2(L)^r & \xrightarrow{\zeta^{(r)}(\alpha)} & W(L). \end{array} \tag{44}$$

By construction, the natural diagram

$$\begin{array}{ccc} \overline{I}_Q^{(r)} & \xrightarrow{\xi^{(r)}} & \mathrm{Inv}_F(\mathrm{Quad}_{2r}, W) \\ \downarrow & & \downarrow \\ \overline{J}_Q^{(r)} & \xrightarrow{\zeta^{(r)}} & \mathrm{Inv}_F(\mathrm{Quad}_2^r, W) \end{array} \tag{45}$$

commutes.

Recall that $\mathrm{Inv}_F(\mathrm{Quad}_{2r}, W)$ is a $W(F)$-module, and through $\Psi_{\overline{\omega}} : \widetilde{W}^{-1}(Q,\gamma) \to W(F)$, it is also a $\widetilde{W}^{-1}(Q,\gamma)$-module.

Lemma 3.6. *The maps $\xi^{(r)}$ and $\zeta^{(r)}$ are morphisms of $\widetilde{W}^{-1}(Q,\gamma)$-modules. They are injective on ${}^0\overline{I}_Q^{(r)}$, ${}^1\overline{I}_Q^{(r)}$, ${}^0\overline{J}_Q^{(r)}$ and ${}^1\overline{J}_Q^{(r)}$.*

Proof. The fact that ξ is a $\widetilde{W}^{-1}(Q,\gamma)$-module morphism follows from the definition: let $\alpha \in \overline{I}_Q^{(r)}$ and $x \in \widetilde{W}^{-1}(Q,\gamma)$. Then for any extension L/F and any $q \in \mathrm{Quad}_{2r}(L)$, if $h \in H_Q^{(r)}(L)$ is such that $(b_{\overline{\omega}_L})_*(h) = q$, we have

$$\begin{aligned} \xi^{(r)}(x\alpha)(q) &= \Phi_{\overline{\omega}_L}(x_L\alpha(h)) \\ &= \Phi_{\overline{\omega}_L}(x_L)\Phi_{\overline{\omega}_L}(\alpha(h)) \\ &= (\Phi_{\overline{\omega}_L}(x))_L\xi^{(r)}(\alpha)(q) \\ &= (x \cdot \xi^{(r)}(\alpha))(q). \end{aligned}$$

If $\xi^{[r)}(\alpha) = 0$, then let K/k be an extension, and let $h \in H_Q^{(r)}(K)$. Then by construction we have

$$\Psi_{\overline{\omega}_{KF}}(\alpha(h)) = \xi^{(r)}(\alpha)((b_{\overline{\omega}_{KF}})_*(h_{KF})) = 0.$$

If α is in ${}^0\overline{I}_Q^{(r)}$ or ${}^1\overline{I}_Q^{(r)}$, then $\alpha(h) \in W(K)/(n_{Q_K})$ or $\alpha(h) \in W^{-1}(Q_K, \gamma_K)$; since $\Psi_{\omega_{KF}}$ is injective on these by Proposition 2.2, $\alpha(h) = 0$, and therefore $\alpha = 0$.

The proofs are completely similar for $\zeta^{[r)}$ so we omit them. $\square$

4 Generators and relations

In this section we give explicit presentations of our modules of invariants, the generators being given by the λ^d.

Precisely, for any $d, r \in \mathbb{N}$, the composition

$$H_{Q_K}^{(r)} \hookrightarrow \widetilde{GW}^{-1}(Q_K, \gamma_K) \xrightarrow{\lambda^d} \widetilde{GW}^{-1}(Q_K, \gamma_K) \to \widetilde{W}^{-1}(Q_K, \gamma_K) \tag{46}$$

for all extensions K/k form an invariant in $I_Q^{(r)}$, which we again denote λ^d. The compatibility with scalar extensions is expressed by the fact that (9) is a commutative diagram of pre-λ-rings.

The image of λ^d in $\overline{I}_Q^{(r)}$ is written $\overline{\lambda}^d$. Note that if d is even then $\lambda^d \in {}^0\overline{I}_Q^{(r)}$, and if d is odd then $\lambda^d \in {}^1\overline{I}_Q^{(r)}$.

4.1 Structure of $\overline{I}_Q^{(r)}$

Lemma 4.1. *Let $r, d \in \mathbb{N}$. The morphism $\xi^{(r)}$ sends $\overline{\lambda}^d \in \overline{I}_Q^{(r)}$ to $\lambda^d \in \mathrm{Inv}_F(Quad_{2r}, W)$.*

Proof. The statement is equivalent to the commutativity of

$$\begin{array}{ccc} H^{(r)}_{Q_F}(L) & \xrightarrow{\lambda^d} & \widetilde{\mathbf{W}}^{-1}(Q_L, \gamma_L) \\ \big\downarrow{\scriptstyle (b_{\overline{\omega}_L})_*} & & \big\downarrow{\scriptstyle \Phi_{\overline{\omega}_L}} \\ \mathrm{Quad}_{2r}(L) & \xrightarrow{\lambda^d} & W(L) \end{array}$$

for all extensions L/F, and this is a simple consequence of the fact that (6) is a commutative diagram of pre-λ-rings. □

The crucial technical result is:

Theorem 4.2. *The $\widetilde{W}^{-1}(Q, \gamma)/(n_Q)$-module $\overline{I}_Q^{(1)}$ is free, with basis $(\overline{\lambda}^0, \overline{\lambda}^1, \overline{\lambda}^2)$.*

Proof. Technically, in order to use our results on generic splitting, we need to distinguish the case where k is quadratically closed, since this is the only case where our description of F using a closed point of degree 2 does not apply. When k is quadratically closed, Q is split, and the result is easily reduced using a Morita equivalence to the case of Witt invariants of Quad_2, which is treated in [2, Thm 27.16]. We now exclude k being quadratically closed for the rest of the proof.

Let $\alpha \in \overline{I}_Q^{(1)}$. Then by [2, Thm 27.16], $\xi^{(1)}(\alpha)$ can be written $q_0 + q_1 \cdot \lambda^1 + q_2 \cdot \lambda^2$ for unique elements $q_i \in W(F)$. We study the residues of the q_i with respect to the valuations coming from closed points of X_Q.

Let us write $K = k(x, y)$, and $KF = K \otimes_k F$. For any field extension E/L, we will write $\rho_{E/L}$ for the scalar extension morphism $W(E) \to W(L)$. Let us consider

$$\begin{aligned} h &= \langle \omega \otimes 1 \rangle_{\gamma_{KF}} \in H^{(1)}_Q(KF) \\ q &= \Psi_{\overline{\omega}_{KF}}(h) \in \mathrm{Quad}_{2r}(KF) \\ \theta &= \rho_{KF/F}(q_0) + q\rho_{KF/F}(q_1) + \langle \det(q) \rangle \rho_{KF/F}(q_2) \in W(KF). \end{aligned}$$

Then using Lemma 4.1, we have

$$\theta = \xi^{(1)}(\alpha)(q) = \Psi_{\overline{\omega}_{KF}}(\alpha(h_{KF})).$$

According to Lemma 1.1, we can write $q = \langle f \rangle \langle\langle g \rangle\rangle$ with

$$\begin{aligned} g &= -\Delta \otimes 1 \in k[x, y] \otimes_k k \subset KF \\ &= ax^2 \otimes 1 + by^2 \otimes 1 - ab \otimes 1 \\ f &= -\mathrm{Trd}_{Q_{KF}}(\omega \otimes \overline{\omega}) \in k[x, y] \otimes k[\overline{x}, \overline{y}] \subset KF \\ &= -2(ax \otimes \overline{x} + by \otimes \overline{y} - ab \otimes 1). \end{aligned}$$

Let v be a valuation on F which is either v_p for some $p \in X_Q^{(1)}$, with valuation ring $\mathcal{O}_v$, uniformizer π and residue field F_v. We can have $p \in Y^{(1)}$ or $p = \infty$.

Then v extends naturally to a discrete rank 1 valuation w on KF with residue field KF_v, trivial on $K \otimes_k k$. If we see KF as the function field of $\mathbb{P}^2 \times X_Q$, then w is the valuation associated to the hypersurface $\mathbb{P}^2 \times \{p\}$.

We write $f = (1 \otimes \pi^l) f'$ with $l \in \mathbb{Z}$ and $f' \in k[x,y] \otimes_k \mathcal{O}_v$. If $p \in Y^{(1)}$ we simply take $l = 0$ and $f' = f$, and if $p = \infty$ we can take $l = -1$. The image of f' in $k[x,y] \otimes_k F_v$ is written $\overline{f}$. Let us take $i \in \{1,2\}$. Then for any $m \in \{0,1,2\}$ we have in $W(KF_v)$:

$$\partial_w^i(q_m) = \rho_{KF_v/F_v}(\partial_v^i(q_0))$$
$$\partial_w^i(\langle f \rangle) = \begin{cases} 0 & \text{if } i = 2 \text{ and } p \in Y^{(1)}, \text{ or } i = 1 \text{ and } p = \infty \\ \langle \overline{f} \rangle & \text{if } i = 1 \text{ and } p \in Y^{(1)}, \text{ or } i = 2 \text{ and } p = \infty \end{cases}$$
$$\partial_w^1(\langle g \rangle) = \langle -\Delta \otimes 1 \rangle$$
$$\partial_w^1(\langle g \rangle) = 0$$

and therefore

$$\partial_w^i(\theta) = \rho_{KF_v/F_v}(\partial_v^i(q_0)) + \langle \overline{f} \rangle \langle\langle -\Delta \otimes 1 \rangle\rangle \rho_{KF_v/F_v}(\partial_v^j(q_1)) \tag{47}$$
$$+ \langle \Delta \otimes 1 \rangle \rho_{KF_v/F_v}(\partial_v^i(q_2)) \in W(KF_v) \tag{48}$$

where $j = i$ if $p \in Y^{(1)}$ and $j \neq i$ if $p = \infty$.

We consider that discrete rank 1 valuation u' on KF_v corresponding to the hypersurface $X_Q \times \{p\}$ in $\mathbb{P}^2 \times \{p\}$, with residue field $F \otimes_k F_v$. Then we take a valuation u'' on $F \otimes_k F_v$ corresponding to any one of the two F_v-rational points in $\{p\} \times \{p\} \simeq \mathrm{Spec}(F_v) \times \mathrm{Spec}(F_v)$. A crucial fact is that if $[f] \in F \otimes_k F_v$ is the image of $\overline{f} \in k[x,y] \otimes_k F_v$, then $u''([f]) = 1$. Indeed, the hypersurface of $Y \times Y$ defined by the image of $f \in k[x,y] \otimes_k k[\overline{x}, \overline{y}]$ in $k[\overline{x}, \overline{y}] \otimes_k k[\overline{x}, \overline{y}] = k[Y] \otimes_k k[Y]$ is the diagonal embedding $Y \to Y \times Y$. Thus the valuation of $[f]$ at any point in the intersection of the diagonal of $X_Q \times X_Q$ and $X_Q \times \{p\}$, which is precisely $\{p\} \times \{p\}$, is 1. This means that if $u = u'' \circ u'$ is the composed valuation on KF_v, with value group $\mathbb{Z}^2$ and residue field F_v, then $u(-\Delta \otimes 1) = (1,0)$ and $u(\overline{f}) = (0,1)$. One also sees that $k \otimes_k F_v \subset \mathcal{O}_u^\times$.

Using $\Delta \otimes 1$ and $\overline{f}$ as uniformizers, we get a residue map

$$\partial_u : W(KF_v) \to W(F_v)[(\mathbb{Z}/2\mathbb{Z})^2]$$

such that, using (47):

$$\partial_u(\partial_w^i(\theta)) = (\partial_v^i(q_0), \partial_v^j(q_1), \partial_v^i(q_2), \partial_v^j(q_1)) \in W(F_v)[(\mathbb{Z}/2\mathbb{Z})^2]. \tag{49}$$

Since θ is in the image of $\Psi_{\varpi_{KF}}$, by the exact sequences (27) and (28) we have that $\partial_w^i(\theta) = 0$ when $i = 2$ and $p \in Y^{(1)}$, which yields

$$\partial_v^2(q_0) = \partial_v^2(q_1) = \partial_v^2(q_2) = 0,$$

in other words $q_0, q_1, q_2 \in W_0(F)$. From this point we assume that $p = \infty$.

Now assume $\alpha \in {}^0\overline{I}_Q^{(r)}$. Then $\alpha(h) \in W(K)/(n_{Q_K})$ so by (27) we have $\partial_w^2(\theta) = 0$, which yields with (49):

$$\partial_v^2(q_0) = \partial_v^1(q_1) = \partial_v^2(q_2) = 0,$$

so $q_0 = \Psi_{\varpi}(\varphi_0)$, $q_1 = \Psi_{\varpi}(h_1)$ and $q_2 = \Psi_{\varpi}(\varphi_2)$, with $\varphi_0, \varphi_2 \in W(k)$ uniquely determined modulo n_Q, and $h_1 \in W^{-1}(Q, \gamma)$ uniquely determined. This shows that

$$\xi^{(1)}(\alpha) = \xi^{(1)}(\varphi_0 + h_1\overline{\lambda}^1 + \varphi_2\overline{\lambda}^2)$$

and by injectivity of $\xi^{(1)}$ on ${}^0\overline{I}_Q^{(r)}$ (Lemma 3.6), we deduce that

$$\alpha = \varphi_0 + h_1\overline{\lambda}^1 + \varphi_2\overline{\lambda}^2,$$

for unique $\varphi_0, \varphi_2 \in W(k)/(n_Q)$ and a unique $h_1 \in W^{-1}(Q, \gamma)$.

Likewise, assume that $\alpha \in {}^1\overline{I}_Q^{(r)}$. This time $\alpha(h) \in W^{-1}(Q_K, \gamma_K)$ so by (27) we have $\partial_w^1(\theta) = 0$, then

$$\partial_v^1(q_0) = \partial_v^2(q_1) = \partial_v^1(q_2) = 0,$$

and again by injectivity of $\xi^{(1)}$ on ${}^1\overline{I}_Q^{(r)}$, we have unique $h_0, h_2 \in W^{-1}(Q, \gamma)$ and a unique $\varphi_1 \in W(k)/(n_Q)$ such that

$$\alpha = h_0 + \varphi_1\overline{\lambda}^1 + h_2\overline{\lambda}^2.$$

In the general case, we write $\alpha = \alpha_0 + \alpha_1$ with $\alpha_0 \in {}^0\overline{I}_Q^{(r)}$ and $\alpha_1 \in {}^1\overline{I}_Q^{(r)}$, and the two previous points show that there are unique $x_0, x_1, x_2 \in \widetilde{W}^{-1}(Q, \gamma)/(n_Q)$ such that

$$\alpha = x_0 + x_1\overline{\lambda}^1 + x_2\overline{\lambda}^2.$$

□

We get the more general case by induction. First:

Corollary 4.3. *For any $r \in \mathbb{N}$, the $\widetilde{W}^{-1}(Q, \gamma)/(n_Q)$-module $\overline{J}_Q^{(r)}$ is free, with basis $(\overline{\lambda}^{\underline{d}})_{\underline{d} \in \{0,1,2\}^r}$, where*

$$\overline{\lambda}^{\underline{d}}(h_1, \dots, h_r) = \prod_{i=1}^{r} \overline{\lambda}^{d_i}(h_i).$$

If $\alpha = \sum_{\underline{d}} x_{\underline{d}}\overline{\lambda}^{\underline{d}}$, then $\alpha \in {}^0\overline{J}_Q^{(r)}$ if and only if $x_{\underline{d}} \in W(k)/(n_Q)$ for all $\underline{d}$ with $|\underline{d}| = \sum_i d_i$ is even, and $x_{\underline{d}} \in W^{-1}(Q, \gamma)$ when $|\underline{d}|$ is odd. Similarly, $\alpha \in {}^1\overline{J}_Q^{(r)}$ if and only if $x_{\underline{d}} \in W^{-1}(Q, \gamma)$ when $|\underline{d}|$ is even and $x_{\underline{d}} \in W(k)/(n_Q)$ when $|\underline{d}|$ is odd.

Proof. This is a direct application of Lemma 3.4, since $\overline{I}_Q^{(1)} = \overline{J}_Q^{(1)}$.

Clearly, $\overline{\lambda}^{\underline{d}} \in {}^0\overline{J}_Q^{(r)}$ when $|\underline{d}|$ is even, and $\overline{\lambda}^{\underline{d}} \in {}^1\overline{J}_Q^{(r)}$ when $|\underline{d}|$ is odd. So if $\alpha = \sum_{\underline{d}} x_{\underline{d}}\overline{\lambda}^{\underline{d}}$, and $x_{\underline{d}} = q_{\underline{d}} + h_{\underline{d}}$, then $\alpha = \alpha_0 + \alpha_1$ with

$$\alpha_0 = \sum_{|\underline{d}| \text{even}} q_{\underline{d}}\overline{\lambda}^{\underline{d}} + \sum_{|\underline{d}| \text{odd}} h_{\underline{d}}\overline{\lambda}^{\underline{d}}$$

and

$$\alpha_1 = \sum_{|\underline{d}| \text{even}} h_{\underline{d}}\overline{\lambda}^{\underline{d}} + \sum_{|\underline{d}| \text{odd}} q_{\underline{d}}\overline{\lambda}^{\underline{d}}$$

and since $\alpha_0 \in {}^0\overline{J}_Q^{(r)}$ and $\alpha_1 \in {}^1\overline{J}_Q^{(r)}$, this is the unique decomposition, and the result follows. □

We can then deduce:

Theorem 4.4. *For any $r \in \mathbb{N}$, the $\widetilde{W}^{-1}(Q,\gamma)/(n_Q)$-module $\overline{I}_Q^{(r)}$ is free, with basis $(\overline{\lambda}^0, \dots, \overline{\lambda}^{2r})$.*

Proof. Let $\alpha \in \overline{I}_Q^{(r)}$, and β its image in $\overline{J}_Q^{(r)}$. We write

$$\beta = \sum_{\underline{d}} x_{\underline{d}} \overline{\lambda}^{\underline{d}}.$$

We can also write

$$\xi^{(r)}(\alpha) = \sum_{d=0}^{2r} y_d \lambda^d$$

for unique elements $y_d \in W(F)$. The image of $\lambda^d \in \mathrm{Inv}_F(\mathrm{Quad}_{2r}, W)$ in $\mathrm{Inv}_F(\mathrm{Quad}_2^r, W)$ is $\sum_{|\underline{d}|=d} \lambda^{\underline{d}}$, which implies by the diagram (45):

$$\zeta^{(r)}(\beta) = \sum_{\underline{d}} \psi_{\overline{\omega}}(x_{|\underline{d}|}) \overline{\lambda}^{\underline{d}}$$

and therefore by uniqueness of the decomposition:

$$x_{\underline{d}} = \Psi_{\overline{\omega}}(x_{|\underline{d}|})$$

for all $\underline{d}$.

If $\alpha \in {}^0\overline{I}_Q^{(r)}$ or $\alpha \in {}^1\overline{I}_Q^{(r)}$, then each $x_{\underline{d}}$ is in $W(k)/(n_Q)$ or $W^{-1}(Q,\gamma)$ by Corollary 4.3, so by injectivity of $\Psi_{\overline{\omega}}$ on these groups (Proposition 2.2), $x_{\underline{d}}$ only depends on $|\underline{d}|$. If $x_{|\underline{d}|}$ is this common value, then

$$\beta = \sum_{d=0}^{2r} x_d \left(\sum_{|\underline{d}|=d} \overline{\lambda}^{\underline{d}} \right).$$

Since $\sum_{|\underline{d}|=d} \overline{\lambda}^{\underline{d}}$ is the image of $\overline{\lambda}^d \in \overline{I}_Q^{(r)}$ in $\overline{J}_Q^{(r)}$, by injectivity of $\overline{I}_Q^{(r)} \to \overline{J}_Q^{(r)}$ we get

$$\alpha = \sum_{d=0}^{2r} x_d \overline{\lambda}^d$$

for unique elements x_d (with $x_d \in W(k)/(n_Q)$ if d is even and $x_d \in W^{-1}(Q,\gamma)$ if d is odd).

Similarly, if $\alpha \in {}^1\overline{I}_Q^{(r)}$ we deduce that $\alpha = \sum_{d=0}^{2r} x_d \overline{\lambda}^d$ for unique elements x_d, with $x_d \in W(k)/(n_Q)$ if d is odd and $x_d \in W^{-1}(Q,\gamma)$ if d is even.

In general, we decompose α as $\alpha_0 + \alpha_1$ with $\alpha_i \in {}^i\overline{I}_Q^{(r)}$ to get the expected result. ☐

Corollary 4.5. *The commutative diagram (37) with exact lines can be refined as*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & n_Q W & \longrightarrow & I_Q^{(r)} & \longrightarrow & \overline{I}_Q^{(r)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & n_Q W & \longrightarrow & J_Q^{(r)} & \longrightarrow & \overline{J}_Q^{(r)} & \longrightarrow & 0
\end{array}$$

Proof. The $n_Q W$ in the diagram come from Proposition 3.5. The surjectivities follows from Theorem 4.4 and Corollary 4.3, since the generators $\overline{\lambda^d} \in \overline{I}_Q^{(r)}$ and $\overline{\lambda^{\underline{d}}} \in \overline{J}_Q^{(r)}$ are the images of $\lambda^d \in I_Q^{(r)}$ and $\lambda^{\underline{d}} \in J_Q^{(r)}$. □

4.2 Structure of $I_Q^{(r)}$

The structure of $I_Q^{(r)}$ is slightly more complicated than that of $I_Q^{(r)}$, as it is not a free module. The (only) obstruction to being free comes from the following simple fact:

Proposition 4.6. *In $I_Q^{(r)}$, for any $d \in \mathbb{N}$ we have that*

$$n_Q \cdot \lambda^d = \begin{cases} 0 & \text{if } d \text{ is odd} \\ \binom{r}{d/2} & \text{if } d \text{ is even.} \end{cases}$$

Proof. The statement when d is odd follows from Corollary 2.3.

When d is even, we first treat the case $r = 1$, where we only need to look at $d = 2$. Let K/k be some extension, and $h = \langle z \rangle_{\gamma_K} \in H_Q^{(1)}(K)$. Then $\lambda^2(h) = \langle \mathrm{Nrd}_{Q_K}(z) \rangle$, and by definition $\mathrm{Nrd}_{Q_K}(z)$ is represented by the 2-fold Pfister form n_{Q_K}, so $n_{Q_K} \lambda^2(h) = n_{Q_K}$.

Now if $r\mathbb{N}^*$, and $h = \langle z_1, \dots, z_r \rangle_\gamma$, we have

$$n_Q \lambda^{2d}(h) = \sum_{d_1 + \dots + d_r = d} n_Q \lambda^{d_1}(\langle z_1 \rangle) \cdots \lambda^{d_1}(\langle z_1 \rangle).$$

In the sum, all d_i are 0, 1 or 2; if one of them is 1 then the term is 0 from the case where d is odd. When all of them are 0 or 2, the case $r = 1$ shows that the term is equal to n_Q. A simple counting argument shows that there are $\binom{r}{d}$ non-zero terms. □

This leads us to introduce

$$\begin{array}{cccc} \chi^{(r)}: & \widetilde{W}^{-1}(Q,\gamma)^{2r+1} & \longrightarrow & \widetilde{W}^{-1}(Q,\gamma) \\ & (x_0, \dots, x_{2r}) & \longmapsto & \sum_{i=0}^{r} \binom{r}{i} x_{2i}. \end{array} \tag{50}$$

We can then state the main result of this article.

Theorem 4.7. *The $\widetilde{W}^{-1}(Q,\gamma)$-module $I_Q^{(r)}$ is generated by $(\lambda^0, \dots, \lambda^{2r})$.*

If $(x_0, \dots, x_{2r}) \in \widetilde{W}^{-1}(Q,\gamma)^{2r+1}$, then the invariant $\alpha = \sum_{d=0}^{2r} x_d \lambda^i$ is constant if and only if $x_d \in n_Q W(k)$ for all $d > 0$, and in that case the constant is $\chi^{(r)}(x_0, \dots, x_{2r})$.

Proof. From the exact sequence of $\widetilde{W}^{-1}(Q,\gamma)$-modules

$$0 \to n_Q W(k) \to I_Q^{(r)} \to \overline{I}_Q^{(r)} \to 0$$

of Corollary 4.5, and the fact that $\overline{I}_Q^{(r)}$ is generated by the $\overline{\lambda}^d$, we deduce that $I_Q^{(r)}$ is generated by the λ^d.

Let $\alpha = \sum_{d=0}^{2r} x_d \lambda^d \in I_Q^{(r)}$, and let $\overline{\alpha}$ be its image in $\overline{I}_Q^{(r)}$. Bu Proposition 3.5, α is constant if and only if $\overline{\alpha}$ is, and since $\overline{I}_Q^{(r)}$ is free and the constant invariants in $\overline{I}_Q^{(r)}$ are the submodule generated by $\overline{\lambda}^0$, $\overline{\alpha}$ is constant if and only if the class of x_d in $\widetilde{W}^{-1}(Q,\gamma)$ is 0 for $d > 0$. By Corollary 2.3, this means that $x_d \in n_Q Q(k)$.

If we write $x_d = n_Q y_d$ for each $d > 0$, then by Proposition 4.6 we have

$$\begin{aligned}\alpha &= x_0 + \sum_{d=1}^{2r} y_d \cdot n_Q \lambda^d \\ &= x_0 + \sum_{i=1}^{r} y_{2i} \cdot n_Q \binom{r}{i} \\ &= \chi^{(r)}(x_0, \ldots, x_{2r}).\end{aligned}$$

□

This theorem gives a presentation of $I_Q^{(r)}$ as a $\widetilde{W}^{-1}(Q,\gamma)$-module:

$$I_Q^{(r)} = \left\langle \lambda^0, \ldots, \lambda^{2r} \,|\, \forall 0 \leqslant i \leqslant r, n_Q \lambda^{2i} = n_Q \binom{r}{i} \lambda^0 \right\rangle. \tag{51}$$

Of course when Q is split $n_Q = 0$ and the module is free.

References

[1] Richard Elman, Nikita Karpenko, and Alexander Merkurjev. *The algebraic and geometric theory of quadratic forms*, volume 58. AMS, 2008.

[2] Skip Garibaldi, Alexander Merkurjev, and Jean-Pierre Serre. *Cohomological Invariants in Galois Cohomology.* AMS, 2003.

[3] Nicolas Garrel. Witt and cohomological invariants of witt classes. *Annals of K-Theory*, 5(2):213–248, 2020.

[4] Nicolas Garrel. Lambda-operations for hermitian forms over algebras with involution. https://arxiv.org/abs/2304.02617, 2022.

[5] Nicolas Garrel. Mixed witt rings of algebras with involution. *Canadian Journal of Mathematics*, 75(2):608–644, 2023.

[6] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The Book of Involutions.* American Mathematical Soc., 1998.

[7] Sean McGarraghy. Exterior powers of symmetric bilinear forms. In *Algebra Colloquium*, volume 9, pages 197–218, 2002.

[8] Anne Quéguiner-Mathieu and Jean-Pierre Tignol. Witt groups of Severi-Brauer varieties and of function fields of conics, April 2023. arXiv:2304.03539 [math].

[9] Donald Yau. *Lambda-rings.* World Scientific Publishing, Singapore ; Hackensack, NJ, 2010.