Mixed Witt rings of algebras with involution of the first kind

Nicolas Garrel

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Introduction

In the 30s, Ernst Witt ([10]) started the algebraic study of quadratic forms over arbitrary fields (we will always assume that fields have characteristic not 2), as opposed to the previous more arithmetic-focused treatments. The key ingredient of his theory is to not only study individual quadratic forms, but consider them as a whole, and give the set $W(K)$ of quadratic forms (up to so-called *Witt equivalence)* over some fixed field K a commutative ring structure, using direct sums and tensor products of quadratic spaces. The algebraic properties of this ring reflect many interesting properties of the underlying field and its quadratic forms (see [6], [9] or [2] for detailed accounts on the theory of the Witt ring of a field). Depending on the situation, it may be more appropriate to consider the Grothendieck-Witt ring $GW(K)$, which is arguably more fundamental (since it reflects isometries of quadratic forms and not only Witt equivalence).

These constructions can be somewhat extended to the framework of ε-hermitian forms over rings with involution (see the reference [4]). If (A, σ) is a ring with involution (we always assume that 2 is invertible), we define the semi-group $SW^\varepsilon(A, \sigma)$ to be the set of isometry classes of regular ε-hermitian modules over (A, σ), with addition given by orthogonal direct sum. Its Grothendieck group is the Grothendieck-Witt group $GW^\varepsilon(A, \sigma)$ of (A, σ), and the quotient by the subgroup of hyperbolic spaces is the Witt group $W^\varepsilon(A, \sigma)$. We retrieve the Witt group of a field K by considering the ring with involution (K, Id) (and taking $\varepsilon = 1$).

One key complication that appears when A is not commutative is that there is no longer a good notion of tensor product of (hermitian) modules over (A, σ), since the tensor product of two modules over A is a module over $A \otimes_K A$, so we are left with only (Grothendieck-)Witt *groups* instead of *rings*. In general there is no obvious remedy to this, but the aim of this article is to show that we can define a natural ring structure when A is an Azumaya algebra. We restrict here to Azumaya algebras over fields, with involutions of the first kind (more general situations will be considered in future articles). The first key observation is that (Grothendieck-)Witt groups are functorial with respect to hermitian Morita equivalences; to properly express this idea, we define a certain monoidal category $\mathbf{Br}_h(K)$ (proposition 1.13), which we call the hermitian Brauer 2-group of K, in which the morphisms are given by hermitian Morita equivalences between algebras with involution.

The second key point is that an algebra with involution is in some sense of order 2 in $\mathbf{Br}_h(K)$: there is a canonical Morita equivalence between $(A \otimes_K A, \sigma \otimes \sigma)$ and (K, Id), given by the so-called involution trace form T_σ (example 1.9). This implies that there is a canonical isomorphism

$$GW(A \otimes_K A, \sigma \otimes \sigma) \xrightarrow{\sim} GW(K).$$

Thus the natural map

$$GW^\varepsilon(A, \sigma) \times GW^\varepsilon(A, \sigma) \to GW(A \otimes_K A, \sigma \otimes \sigma)$$

given by the tensor product over K actually defines a map

$$GW^\varepsilon(A, \sigma) \times GW^\varepsilon(A, \sigma) \to GW(K).$$

Using the fact that $GW(K)$ is a ring and $GW^\varepsilon(A, \sigma)$ is a $GW(K)$-module, this allows us to construct a $\mathbb{Z}/2\mathbb{Z}$-graded commutative ring

$$\widetilde{GW}^\varepsilon(A, \sigma) = GW(K) \oplus GW^\varepsilon(A, \sigma).$$

If one wants to work with involutions of both types (orthogonal and symplectic) at once, it is much more convenient to consider all at once hermitian *and* anti-hermitian forms over (A, σ), and hence rather construct what we call the mixed Grothendieck-Witt ring of (A, σ) (proposition 2.12):

$$\widetilde{GW}(A, \sigma) = GW(K) \oplus GW^{-1}(K) \oplus GW(A, \sigma) \oplus GW^{-1}(A, \sigma),$$

as well as its quotient the mixed Witt ring:

$$\widetilde{W}(A, \sigma) = W(K) \oplus W(A, \sigma) \oplus W^{-1}(A, \sigma),$$

which are commutative rings, naturally graded over the Klein group, and are functorial with respect to the category $\mathbf{Br}_h(K)$. The associativity, commutativity and functoriality of these rings can be deduced from properties of $\mathbf{Br}_h(K)$, which we study in the first section, most importantly the fact that this category is *strongly symmetric* (corollary 1.24), and has *strong 2-torsion* (theorem 1.30). The second section is then dedicated to the definition of the mixed rings, and the study of their basic properties. More advanced investigations will be featured in future articles, including the λ-ring structure of $\widetilde{GW}(A, \sigma)$, the fundamental filtration of $\widetilde{W}(A, \sigma)$ and its connections to cohomology, and the prime spectrum of $\widetilde{W}(A, \sigma)$ and its relation with signatures of hermitian forms.

Preliminaries and conventions

We fix a base field K of characteristic not 2, and we identify symmetric bilinear forms and quadratic forms over K, through $b \mapsto q_b$ with $q_b(x) = b(x, x)$. Diagonal quadratic forms are denoted $\langle a_1, \dots, a_n \rangle$, with $a_i \in K^*$, and $\langle\langle a_1, \dots, a_n \rangle\rangle$ is the n-fold Pfister form $\langle 1, -a_1 \rangle \cdots \langle 1, -a_n \rangle$. We always assume that bilinear forms are non-degenerated.

All rings are associative and with unit, and ring morphisms preserve the units. The group of invertible elements of a ring A is denoted $A^\times$. Unless otherwise specified, modules are by default modules on the *right*, and are assumed to be faithful modules. Every K-algebra and every module over such an algebra have finite dimension over K. If A is a K-algebra, and V is a right A-module, then $\mathrm{End}_A(V)$ is a K-algebra acting on V on the left, with the tautological action of functions on V. On the other hand, if V is a A-module on the *left*, then we endow $\mathrm{End}_A(V)$ with the product *opposite* to usual function composition, so that $\mathrm{End}_A(V)$ acts on V on the *right*. If A and B are K-algebras, a B-A-bimodule is always supposed to be over K, meaning that the right and left actions of K on V coincide. If A is a central simple algebra over K, we write $\mathrm{Trd}_A : A \to K$ for the reduced trace of A.

When we say that (A, σ) is an algebra with involution over K, we mean that A is a central simple algebra over K, and that σ is an involution of the first kind on A, so σ is an anti-automorphism of K-algebra of A, with $\sigma^2 = \mathrm{Id}_A$. In general, "involution" will be synonym with "involution of the first kind". If $\varepsilon = \pm 1$, we define $\mathrm{Sym}^\varepsilon(A, \sigma)$ as the set of ε-symmetric elements of A, which satisfy $\sigma(a) = \varepsilon a$. We also write $\mathrm{Sym}^\varepsilon(A^\times, \sigma)$ for the set of invertible ε-symmetric elements. Recall that the involution σ is orthogonal if $\dim_K(\mathrm{Sym}(A, \sigma)) = n(n+1)/2$, and it is symplectic if $\dim_K(\mathrm{Sym}(A, \sigma)) = n(n-1)/2$. In particular, (K, Id) is an algebra with orthogonal involution. A quaternion algebra admits a unique symplectic involution, called its canonical involution.

If L/K is any field extension, and X is an object (algebra, module, involution, hermitian form, etc.) over K, then X_L is the corresponding object over L, obtained by base change.

A semi-group is a set endowed with an associative binary product (so the difference with a monoid is the existence of a unit). If Γ is a set, then a semi-group S is Γ-graded if it is equipped with a direct sum decomposition $S = \bigoplus_{g\in\Gamma} S_g$. The Grothendieck group $G(S)$ of S is the universal solution to the problem of finding a morphism $S \to G$ where G is a group. Since the functor $S \mapsto G(S)$ preserves direct sums, if S is Γ-graded then $G(S)$ inherits a natural Γ-grading.

What we call a semi-ring is a triple $(S, +, \cdot)$ such that $(S, +)$ is a commutative semi-group, $(S, \cdot)$ is a monoid, and we have the distributive law (often in the literature it is asked that $(S, +)$ is a monoid). If Γ is a semi-group (written multiplicatively), we say that S is Γ-graded if $(S, +)$ has a Γ-grading such that $S_g S_h \subset S_{gh}$ for all $g, h \in \Gamma$. If S is a semi-ring then $G(S)$ is naturally a ring, and if S is Γ-graded then $G(S)$ is a Γ-graded ring.

If $(C, \otimes, I)$ is a monoidal category (see [8, VII.1] for the definitions), we will always write as though it were *strictly* monoidal, which means that we will not explicitly write the associators and unitors when we technically should. In particular, we use notations such as $x^{\otimes n}$ for any $n \in \mathbb{N}$, with $x^{\otimes 0} = I$. This is harmless according to MacLane's coherence theorem. All monoidal functors are assumed to be strong, and they send the unit object to the unit object (which can always be assumed up to a natural isomorphism).

1 The hermitian Brauer 2-group

In this section we review hermitian Morita theory, as developed in [3] or [4], in the case of algebras with involution (for which we take [5] as a reference). We adopt a categorical point of view that allows the theory to be expressed in a very efficient way. The idea that (non-hermitian) Morita theory can be expressed as the definition of some 2-category of algebras and bimodules has been explored for instance in [1], but as far as we know this is the first time the hermitian analogue is written down explicitly (though for simplicity we only define a 1-category).

In fact, we define a certain category $\mathbf{Br}_h(K)$, which we call the *hermitian Brauer 2-group* of the field K, and much of the classical hermitian Morita theory (at least for central simple algebras with involution of the first kind) is contained in the fact that $\mathbf{Br}_h(K)$ is a groupoid. Furthermore, the special case we consider here has remarkable features of symmetry (see proposition 1.13 and theorem 1.30) which are crucial for the construction of the mixed Witt ring in the second section.

1.1 Hermitian modules and involutions

We start, for reader's convenience as well as for establishing notations, by reviewing basic facts about hermitian modules (see [5] for a reference, or [4] for an account over general rings with involution).

Let A and B be central simple algebras over K. We say that a B-A-bimodule V is a Morita bimodule if the following equivalent conditions hold (see [5, 1.10]):

- the left action of B gives an isomorphism $B \simeq \mathrm{End}_A(V)$;
- the right action of A gives an isomorphism $A \simeq \mathrm{End}_B(V)$.

Then A and B are Brauer-equivalent iff there exists a Morita B-A-bimodule, which is then unique up to isomorphism.

Example 1.1. We can always see A as a tautological A-A-bimodule, and it is a Morita bimodule. We will often write $|A|$ when we see A as a vector space or a module.

Example 1.2. It is a defining property of Azumaya algebras that the natural "sandwich" map

$$\begin{array}{rcl} A \otimes_K A^{op} & \longrightarrow & \mathrm{End}_K(|A|) \\ a \otimes b & \longmapsto & (x \mapsto axb) \end{array} \tag{1}$$

is a K-algebra isomorphism, so $|A|$ is a Morita $(A \otimes_K A^{op})$-K-bimodule. In general, a Morita B-A-bimodule is the same thing as a Morita $(B \otimes_K A^{op})$-K-bimodule.

Recall that if (A, σ) is an algebra with involution over K and $\varepsilon = \pm 1$, a ε-hermitian module (V, h) over (A, σ) is a (right) A-module V equipped with a ε-hermitian form h, meaning that

$$h : V \times V \to A$$

is bi-additive and satisfies for all $x, y \in V$ and all $a, b \in A$:

$$h(xa, yb) = \sigma(a)h(x, y)b,$$
$$h(y, x) = \varepsilon\sigma(h(x, y)).$$

We always assume that ε-hermitian forms are regular, meaning that the induced map $V \to \mathrm{Hom}_A(V, A)$ given by $x \mapsto h(x, -)$ is bijective. An isometry between two ε-hermitian modules is a module isomorphism which preserves the hermitian forms.

Example 1.3. A ε-hermitian module over (K, Id) is simply a ε-symmetric bilinear module over K.

Example 1.4. Let (A, σ) be an algebra with involution over K, and let $a \in \mathrm{Sym}^{\varepsilon}(A^{\times}, \sigma)$ (for instance, $a \in K^{\times}$ and $\varepsilon = 1$). Then we define a ε-hermitian form over $|A|$, by:

$$\begin{array}{rrcl} \langle a \rangle_{\sigma} : & |A| \times |A| & \longrightarrow & A \\ & (x, y) & \longmapsto & \sigma(x) a y. \end{array}$$

We call such a form *elementary diagonal*. We will write $\langle a_1, \dots, a_n \rangle_{\sigma}$ for an orthogonal sum $\langle a_1 \rangle_{\sigma} \perp \cdots \perp \langle a_n \rangle_{\sigma}$ (where all a_i are ε-symmetric), and call such a form *diagonal*.

Remark 1.5. If A is a division algebra, then any ε-hermitian form is diagonal in this sense (up to isometry), but this is not the case in general.

If (V_i, h_i) is a ε_i-hermitian module over an algebra with involution (A_i, σ_i) (for $i \in \{1, 2\}$) then $(V_1 \otimes_K V_2, h_1 \otimes h_2)$ is naturally a $\varepsilon_1\varepsilon_2$-hermitian module over $(A_1 \otimes_K A_2, \sigma_1 \otimes \sigma_2)$.

Example 1.6. Let (A, σ) and (B, τ) be two algebras with involution over K, and let $a \in \mathrm{Sym}^{\varepsilon}(A^{\times}, \sigma)$ and $b \in \mathrm{Sym}^{\varepsilon'}(B^{\times}, \tau)$. Then $(|A| \otimes_K |B|, \langle a \rangle_{\sigma} \otimes \langle b \rangle_{\tau})$ is a $\varepsilon\varepsilon'$-hermitian module over $(A \otimes_K B, \sigma \otimes \tau)$, which is naturally isometric to $(|A \otimes_K B|, \langle a \otimes b \rangle_{\sigma \otimes \tau})$.

If (V, h) is a ε-hermitian module over (A, σ), then the adjoint involution σ_h on $\mathrm{End}_A(V)$ is defined (see [5, 4.1]) by the fact that for all $x, y \in V$ and all $f \in \mathrm{End}_A(V)$:

$$h(f(x), y) = h(x, \sigma_h(f)(y)).$$

If (A, σ) and (B, τ) are algebras with involution over K, we say that (V, h) is a Morita ε-hermitian bimodule between (B, τ) and (A, σ) if V is a Morita B-A-bimodule, h is a ε-hermitian form over (A, σ) on V, and τ corresponds to σ_h through the natural isomorphism $B \simeq \mathrm{End}_A(V)$. In particular, any ε-hermitian module (V, h) over (A, σ) is naturally a Morita ε-hermitian bimodule between $(\mathrm{End}_A(V), \sigma_h)$ and (A, σ). An isomorphism of Morita ε-hermitian bimodules is a bimodule isomorphism which is also an isometry.

If V is any Morita B-A-bimodule, then there always exists a ε-hermitian form h on V such that (V, h) is a Morita ε-hermitian bimodule between (B, τ) and (A, σ) (see [5, 4.2]). If σ and τ have the same type, then we must take $\varepsilon = 1$, otherwise $\varepsilon = -1$. Moreover, if h' is any other choice, then there is $\lambda \in K^{\times}$ such that $h' = \langle \lambda \rangle h$.

Example 1.7. If $a \in \mathrm{Sym}^{\varepsilon}(A^{\times}, \sigma)$, then $(|A|, \langle a \rangle_{\sigma})$ is a Morita ε-hermitian bimodule between (A, σ_a) and (A, σ), where $\sigma_a(x) = a^{-1}\sigma(x)a$. In particular, if $a \in K^{\times}$ then $\sigma_a = \sigma$, so $(|A|, \langle a \rangle_{\sigma})$ is a Morita hermitian bimodule between (A, σ) and itself.

Example 1.8. If (V_i, h_i) is a Morita ε_i-hermitian bimodule between (B_i, τ_i) and (A_i, σ_i) (for $i = 1, 2$), then $(V_1 \otimes_K V_2, h_1 \otimes h_2)$ is a Morita $\varepsilon_1\varepsilon_2$-hermitian module between $(B_1 \otimes_K B_2, \tau_1 \otimes \tau_2)$ and $(A_1 \otimes_K A_2, \sigma_1 \otimes \sigma_2)$.

Example 1.9. Using the involution σ on A, we can twist the usual sandwich map (1) to

$$\begin{array}{rcl} A \otimes_K A & \longrightarrow & \mathrm{End}_K(|A|) \\ a \otimes b & \longmapsto & (x \mapsto ax\sigma(b)). \end{array} \tag{2}$$

We call this action of $A \otimes_K A$ on $|A|$ the "twisted sandwich action" (relative to σ). We will write $|A|_\sigma$ instead of $|A|$ when we see it as a left $A \otimes_K A$-module with this action. It is shown in [5, 11.1] that the so-called involution trace form

$$\begin{array}{rrcl} T_\sigma : & |A| \times |A| & \longrightarrow & K \\ & (x,y) & \longmapsto & \mathrm{Trd}_A(\sigma(x)y) \end{array} \tag{3}$$

is a symmetric bilinear form of the K-vector space $|A|$, such that $(|A|_\sigma, T_\sigma)$ is a Morita hermitian bimodule between $(A \otimes_K A, \sigma \otimes \sigma)$ and (K, Id).

If (V, h) is a Morita ε-hermitian bimodule between (B, τ) and (A, σ), then we define $(\overline{V}, \overline{h})$, where $\overline{V}$ is the A-B-bimodule defined as V as an additive group, with the action

$$a \cdot v \cdot b = \tau(b) \cdot v \cdot \sigma(a),$$

and $\overline{h} : \overline{V} \times \overline{V} \to B$ is characterized by

$$\overline{h}(x,y)z = xh(y,z)$$

for all $x, y, z \in V$. Then it is easy to see that $(\overline{V}, \overline{h})$ is a Morita ε-hermitian bimodule between (A, σ) and (B, τ).

1.2 The monoidal category $\mathbf{Br}_h(K)$

We now define a category $\mathbf{Br}_h(K)$, which we call the *hermitian Brauer 2-group* of K, such that:

- the objects are the algebras with involution (A, σ) over K;
- the morphisms from (B, τ) to (A, σ) are the isomorphism classes of Morita ε-hermitian bimodules between (B, τ) and (A, σ).

We will usually identify a ε-hermitian bimodule and its isomorphism class when no confusion is possible. If $f : (B, \tau) \to (A, \sigma)$ is a morphism in $\mathbf{Br}_h(K)$, we will sometimes write (V_f, h_f) for the corresponding ε-hermitian bimodule. Conversely, given (V, h), we sometimes write $f_{(V,h)}$, or simply f_h, for the corresponding morphism in $\mathbf{Br}_h(K)$.

Definition 1.10. *If (V, h) is a ε-hermitian module, we say that ε is the* sign *of h, and we denote it by ε_h. Let $f : (B, \tau) \to (A, \sigma)$ be a morphism in $\mathbf{Br}_h(K)$. We define the sign ε_f of f as the sign of h_f.*

We also define the type of f as $t(f) = 1$ if τ is orthogonal, and $t(f) = -1$ if τ is symplectic.

To properly define $\mathbf{Br}_h(K)$ as a category, we need to specify how to compose morphisms.

Proposition-definition 1.11. *Let (A,σ), (B,τ) and (C,θ) be algebras with involution over K. Let (U,h) be a ε-hermitian Morita bimodule between (C,θ) and (B,τ), and let (V,h') be a ε'-hermitian Morita bimodule between (B,τ) and (A,σ).*

Then $(V,h')\circ(U,h) = (V\circ U, h'\circ h)$ is a $\varepsilon\varepsilon'$-hermitian Morita bimodule between (C,θ) and (A,σ), where

$$V\circ U = U\otimes_B V$$

and

$$(h'\circ h)(u\otimes v, u'\otimes v') = h'(v, h(u,u')v'). \tag{4}$$

Proof. This is a direct consequence of [4, I.8.1] in the special case of central simple algebras with involutions of the first kind. □

Example 1.12. Let $f:(B,\tau)\to(A,\sigma)$ be a morphism in $\mathbf{Br}_h(K)$, and let $b\in \mathrm{Sym}^{\varepsilon}(B^\times,\tau)$, $a\in\mathrm{Sym}^{\varepsilon'}(A^\times,\sigma)$. Then the underlying space of $\langle a\rangle_\sigma\circ f\circ\langle b\rangle_\tau$ is $|B|\otimes_B V\otimes_A|A|$, which is canonically identified with V, and the $\varepsilon\varepsilon_f\varepsilon'$-hermitian form is:

$$\begin{array}{ccc} V\times V & \longrightarrow & A \\ (x,y) & \longmapsto & h(xa,by). \end{array}$$

In particular, $f\circ\langle 1\rangle_\tau = \langle 1\rangle_\sigma\circ f = f$.

We can then state:

Proposition 1.13. *With the composition of morphisms being given by definition 1.11, $\mathbf{Br}_h(K)$ is a category, and the identity of (A,σ) is $(|A|,\langle 1\rangle_\sigma)$. Moreover, $\mathbf{Br}_h(K)$ is actually a groupoid, and the inverse of a morphism (V,h) is $(\overline{V},\varepsilon_h\overline{h})$.*

Proof. All statements are reformulations of classical statements in hermitian Morita theory, in the special case of central simple algebras with involution of the first kind. The associativity of the composition is proved in [4, lemma I.8.1.1]. The statement on identities follows from example 1.12, and the statement about inverses is proved in [4, prop I.9.3.4]. □

Remark 1.14. It would have been natural to define instead a 2-category, where the 2-morphisms are the isomorphisms of hermitian bimodules; what we defined is the 1-truncature of this 2-category. But we do not need such a level of complexity in the structure for our constructions, so we stick with plain categories.

Remark 1.15. If f is a morphism in $\mathbf{Br}_h(K)$, then we have defined in 1.10 both its *sign* and its *type*. For composable morphisms, the sign of $f\circ g$ is the product of the signs of f and g, while the type of $f\circ g$ is simply the type of g.

The tensor product of algebras with involution and of ε-hermitian modules naturally makes $\mathbf{Br}_h(K)$ into a monoidal category, with unit object (K,Id). The various coherence axioms follow directly from the usual properties of tensor products (we leave the easy proofs to the reader).

Remark 1.16. For any category C, let us write $\pi_0(C)$ for its set of isomorphism classes. When C is monoidal, $\pi_0(C)$ inherits a natural monoid structure. As we have already stated, there is a morphism in $\mathbf{Br}_h(K)$ between (B,τ) and (A,σ)

iff A and B are Brauer-equivalent. Since moreover A admits an involution (of the first kind) iff $2[A] = 0$ in $\mathrm{Br}(K)$, we see that there is a natural identification between $\pi_0(\mathbf{Br}_h(K))$ and the 2-torsion $\mathrm{Br}(K)[2]$ of the Brauer group of K, which is a monoid isomorphism. In particular, $\pi_0(\mathbf{Br}_h(K))$ is a 2-torsion group, which is also illustrated by example 1.9.

1.3 Automorphisms and isometries

We defined a category $\mathbf{Br}_h(K)$ in which the (iso)morphisms between two algebras with involution correspond to hermitian Morita equivalences. But obviously there is a more elementary notion of isomorphism between algebras with involution. Precisely, we may define a category $\mathbf{AlgInv}(K)$, with the same objects as $\mathbf{Br}_h(K)$, but where the morphisms are K-algebra isomorphisms which are compatible with the involutions.

Then to any algebra isomorphism $\varphi : (B,\tau) \to (A,\sigma)$ we can associate the Morita hermitian bimodule $(|A|_\varphi, \langle 1\rangle_\sigma)$ between (B,τ) and (A,σ), where $|A|_\varphi = |A|$ as a right A-module, and the left action of B is given by $b\cdot a = \varphi(b)a$ for any $b \in B$ and $a \in |A|$. It is easily seen that this is indeed a Morita hermitian bimodule, and that this defines a functor

$$\Theta : \mathbf{AlgInv}(K) \to \mathbf{Br}_h(K) \tag{5}$$

which is the identity on objects.

Obviously this functor is far from being full, since there are isomorphisms in $\mathbf{Br}_h(K)$ between (B,τ) and (A,σ) whenever A and B are Brauer-equivalent. On the other hand, we can look at the situation when (A,σ) and (B,τ) are actually isomorphic as algebras with involution, and we are readily reduced to the case where they are equal. We call

$$\mathrm{Aut}_{\mathcal{M}}(A,\sigma) := \mathrm{Aut}_{\mathbf{Br}_h(K)}(A,\sigma)$$

the group of Morita automorphisms of (A,σ), and

$$\mathrm{Aut}_K(A,\sigma) := \mathrm{Aut}_{\mathbf{AlgInv}(K)}(A,\sigma)$$

the group of algebraic automorphisms of (A,σ). We also say that a morphism in $\mathbf{Br}_h(K)$ is algebraic if it is in the image of Θ. Then we have a group morphism

$$\Theta : \mathrm{Aut}_K(A,\sigma) \longrightarrow \mathrm{Aut}_{\mathcal{M}}(A,\sigma)$$

and we want to understand the groups involved, as well as the kernel and cokernel.

Recall from [5, 12.14] that the subgroup $\mathrm{Sim}(A,\sigma) \subset A^\times$ of *similitudes* of (A,σ) is defined as

$$\mathrm{Sim}(A,\sigma) = \{a \in A \mid a\sigma(a) \in K^\times\}.$$

When $a \in \mathrm{Sim}(A,\sigma)$, we can define its *multiplier* $\mu(a) = a\sigma(a) = \sigma(a)a \in K^\times$, and μ is a group morphism $\mathrm{Sim}(A,\sigma) \to K^\times$. Then $\mathrm{Ker}(\mu)$ is the subgroup $\mathrm{Iso}(A,\sigma)$ of *isometries* of (A,σ), and $\mathrm{Im}(\mu) = G(A,\sigma)$ is the group of multipliers of (A,σ). The following proposition gives a complete picture of the situation:

Proposition 1.17. *Let (A, σ) be an algebra with involution over K. Then we have a commutative diagram of groups with exact rows and columns:*

$$
\begin{array}{ccccccccccc}
 & & 1 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & K^\times & \xrightarrow{\cdot 2} & K^{\times 2} & \longrightarrow & 1 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Iso}(A,\sigma) & \longrightarrow & \mathrm{Sim}(A,\sigma) & \xrightarrow{\mu} & K^\times & \longrightarrow & K^\times/G(A,\sigma) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{int}} & & \downarrow{\scriptstyle \mathrm{int}} & & \downarrow{\scriptstyle \lambda\mapsto(|A|,\langle\lambda\rangle_\sigma)} & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Ker}(\Theta) & \longrightarrow & \mathrm{Aut}_K(A,\sigma) & \xrightarrow{\Theta} & \mathrm{Aut}_{\mathcal{M}}(A,\sigma) & \longrightarrow & \mathrm{Coker}(\Theta) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & & 1 & &
\end{array}
$$

In particular, there is a canonical isomorphism $\mathrm{Aut}_{\mathcal{M}}(A, \sigma) \simeq K^\times/K^{\times 2}$.

Proof. To prove that the diagram commutes, the only non-trivial thing to show is that for any $a \in \mathrm{Sim}(A, \sigma)$, $\Theta(\mathrm{int}(a)) \simeq (|A|, \langle\mu(a)\rangle_\sigma)$. By definition, $\Theta(\mathrm{int}(a))$ corresponds to the hermitian A-A-bimodule $(|A|_a, \langle 1\rangle_\sigma)$, where $|A|_a = |A|$ as a right A-module, and the action on the left is given by $x \cdot y = axa^{-1}y$ for all $x \in A$, $y \in |A|_a$. Then we easily see that

$$
\begin{array}{cccc}
f: & |A|_a & \longrightarrow & |A| \\
 & y & \longmapsto & a^{-1}y
\end{array}
$$

is a bimodule isomorphism. Since

$$\langle\mu(a)\rangle_\sigma(f(x), f(y)) = \sigma(f(x))\mu(a)f(y) = \mu(a)\sigma(x)\sigma(a)^{-1}a^{-1}y = \sigma(x)y,$$

the map f induces an isometry from $\langle 1\rangle_\sigma$ to $\langle\mu(a)\rangle_\sigma$, which shows that the diagram commutes.

The exactness of each row is clear. The exactness of the second column is a consequence of [5, 12.15]. For the third column, the surjectivity comes from the fact that any automorphism of (A, σ) in $\mathbf{Br}_h(K)$ has the form $(|A|, h)$, and since $h = \langle 1\rangle_\sigma$ is a possible choice, any other choice must have the form $h = \langle\lambda\rangle_\sigma$. To prove exactness at $K^\times$, we must also show that $(|A|, \langle\lambda\rangle_\sigma)$ and $(|A|, \langle 1\rangle_\sigma)$ are isomorphic iff λ is a square in $K^\times$. But any bimodule isomorphism $f : |A| \to |A|$ has the form $f(x) = ax$ for some $a \in K^\times$, so if it induces an isometry from $\langle\lambda\rangle_\sigma$ to $\langle 1\rangle_\sigma$, we must have $\lambda = a^2$.

For the first and fourth columns, the only thing to show is the surjectivity, but it is easily obtained by a diagram chase now that we have shown exactness everywhere else. □

Remark 1.18. Given two morphisms $f_i : (B_i, \tau_i) \to (A, \sigma)$ in $\mathbf{Br}_h(K)$, since $\mathbf{Br}_h(K)$ is a groupoid there is a unique $g : (B_1, \tau_1) \to (B_2, \tau_2)$ such that $f_1 = f_2 \circ g$. Now h_{f_1} is isometric to h_{f_2} iff g is algebraic. Thus $\mathbf{Br}_h(K)$ cannot detect isometric modules on its own, but it can if we add the data of Θ.

1.4 Symmetries and the Goldman element

It is easy to see that the functor $\Theta : \mathbf{AlgInv}(K) \to \mathbf{Br}_h(K)$ is a (strict) monoidal functor, by definition of the monoidal structure of $\mathbf{Br}_h(K)$. But by the usual properties of tensor products, $\mathbf{AlgInv}(K)$ is more than just a monoidal category, it is a *symmetric* monoidal category (see [8, VII.7] for the definitions).

Proposition 1.19. *The monoidal category* $\mathbf{Br}_h(K)$ *is symmetric if we consider the switch map*

$$(A,\sigma)\otimes_K (B,\tau) \xrightarrow{\Theta(s)} (B,\tau)\otimes_K (A,\sigma),$$

where $s : (A,\sigma)\otimes_K (B,\tau) \to (B,\tau)\otimes_K (A,\sigma)$ *is the switch map in* $\mathbf{AlgInv}(K)$.

Proof. Since the switch maps, the associators, and the unit maps in $\mathbf{Br}_h(K)$ are all images of morphisms in $\mathbf{AlgInv}(K)$ by Θ, all conditions can be verified in $\mathbf{AlgInv}(K)$. □

In particular, this means that for any algebra with involution (A,σ) over K and any $n \in \mathbb{N}$ there is a natural group morphism

$$\mathfrak{S}_n \longrightarrow \mathrm{Aut}_K(A^{\otimes n},\sigma^{\otimes n}) \xrightarrow{\Theta} \mathrm{Aut}_{\mathcal{M}}(A^{\otimes n},\sigma^{\otimes n}).$$

A remarkable feature of Azumaya algebras is the existence of the so-called Goldman element (see [5, 3.A]).

Definition 1.20. *Let A be a central simple algebra over K. Its Goldman element $g_A \in A\otimes_K A$ is defined by the fact that the sandwich map (1) sends g_A, seen as an element of $A\otimes_K A^{op}$ to the reduced trace* $\mathrm{Trd}_A : A \to K$, *viewed as a linear map* $|A| \to |A|$.

It is shown in [5, 10.1] that the natural morphism $\mathfrak{S}_n \to \mathrm{Aut}_K(A^{\otimes n})$ admits a lift

$$\begin{array}{ccc} & & (A^{\otimes n})^\times \\ & \nearrow & \downarrow \mathrm{int} \\ \mathfrak{S}_n & \longrightarrow & \mathrm{Aut}_K(A^{\otimes n}) \end{array}$$

where $\mathfrak{S}_n \to (A^{\otimes n})^\times$ is uniquely charaterized by

$$(i,i+1) \mapsto 1\otimes\cdots\otimes g_A \otimes 1 \otimes \cdots \otimes 1$$

(with g_A occupying the i and $i+1$ slots).

Proposition 1.21. *Let (A,σ) be an algebra with involution over K. Then the natural morphism $\mathfrak{S}_n \to (A^{\otimes n})^\times$ gives a commutative diagram of groups:*

$$\begin{array}{ccc} & & \mathrm{Iso}(A^{\otimes n},\sigma^{\otimes n}) \\ & \nearrow & \downarrow \mathrm{int} \\ \mathfrak{S}_n & \longrightarrow & \mathrm{Aut}_K(A^{\otimes n},\sigma^{\otimes n}). \end{array}$$

Proof. The only thing to show is that $\mathfrak{S}_n \to (A^{\otimes n})^\times$ actually takes values in $\mathrm{Iso}(A^{\otimes n},\sigma^{\otimes n})$. It is enough to consider transpositions of the form $(i,i+1)$, which means that it is enough to show that $g_A \in \mathrm{Iso}(A^{\otimes 2},\sigma^{\otimes 2})$. Since $g_A^2 = 1$ ([5, 3.6]), this is the same as proving that g_A is symmetric for $\sigma\otimes\sigma$, which is proved in [5, 10.19]. □

This leads us to the following definition:

Definition 1.22. *Let $(C, \otimes, I)$ be a symmetric monoidal category. We say that C is strongly symmetric if for any object $c \in C$, the switch map $x \otimes x \to x \otimes x$ is the identity.*

Example 1.23. If M is a set, we write $\langle M \rangle$ for the discrete category with underlying set M. If M is a monoid then $\langle M \rangle$ is naturally a monoidal category, which is symmetric iff M is commutative. When M is commutative, $\langle M \rangle$ is actually strongly symmetric.

Of course this is motivated by:

Corollary 1.24. *The symmetric monoidal category $\mathbf{Br}_h(K)$ is strongly symmetric.*

Proof. The morphism $\mathfrak{S}_n \to \mathrm{Aut}_{\mathcal{M}}(A^{\otimes n}, \sigma^{\otimes n})$ factors through $\mathrm{Iso}(A^{\otimes n}, \sigma^{\otimes n})$ by proposition 1.21, which by proposition 1.17 means that it is trivial. In particular, for $n = 2$, this means that the switch map is trivial. □

We can give a few characterizations of strongly symmetric categories. If C and D are two monoidal categories, we write $\mathrm{Hom}_{\otimes}(C, D)$ for the category of monoidal functors $C \to D$, with monoidal natural transformations as morphisms. If C and D are symmetric, we define $\mathrm{Hom}^s_{\otimes}(C, D)$ as the full subcategory of $\mathrm{Hom}_{\otimes}(C, D)$ consisting of symmetric monoidal functors.

For any monoidal category, we have a canonical functor

$$\mathrm{Hom}_{\otimes}(\langle \mathbb{N} \rangle, C) \longrightarrow C$$

sending $F : \langle \mathbb{N} \rangle \to C$ to $F(1)$, and this functor is an equivalence of category, with inverse the canonical functor $C \to \mathrm{Hom}_{\otimes}(\langle \mathbb{N} \rangle, C)$ which sends an object x to the monoidal functor $r \mapsto x^{\otimes r}$. This is of course analogous to the fact that any monoid M is in bijection with the monoid morphisms $\mathbb{N} \to M$.

Proposition 1.25. *Let $(C, \otimes, I)$ be a symmetric monoidal category. Then the following conditions are equivalent:*

(i) *C is strongly symmetric;*

(ii) *for any object $x \in C$, the the natural group morphism $\mathfrak{S}_2 \to \mathrm{Aut}_C(x^{\otimes 2})$ is trivial;*

(iii) *for any object $x \in C$ and any $n \in \mathbb{N}^*$, the the natural group morphism $\mathfrak{S}_n \to \mathrm{Aut}_C(x^{\otimes n})$ is trivial;*

(iv) $\mathrm{Hom}_{\otimes}(\langle \mathbb{N} \rangle, C) = \mathrm{Hom}^s_{\otimes}(\langle \mathbb{N} \rangle, C)$*;*

(v) *the canonical functor* $\mathrm{Hom}^s_{\otimes}(\langle \mathbb{N} \rangle, C) \to C$ *is an equivalence of categories.*

Proof. The equivalence between the first three conditions is clear, using that $\mathfrak{S}_n$ is generated by transpositions. For the equivalence between (iv) and (v), since $\mathrm{Hom}_{\otimes}(\langle \mathbb{N} \rangle, C) \to C$ is an equivalence, then its restriction $\mathrm{Hom}^s_{\otimes}(\langle \mathbb{N} \rangle, C) \to C$ is an equivalence iff every monoidal functor $\langle \mathbb{N} \rangle \to C$ is monoidally isomorphic to a symmetric functor; but this means it is already symmetric. Finally, it is clear that for any $x \in C$, the functor $x \mapsto x^{\otimes r}$ is symmetric iff the switch map $x \otimes x \to x \otimes x$ is the identity, which gives the equivalence between (i) and (iv). □

1.5 Strong torsion

We can adapt the characterization of proposition 1.25 to treat torsion. The motivation is the following: as we stated in the introduction, we want to use the isomorphism $(A \otimes_K A, \sigma \otimes \sigma) \to (K, \mathrm{Id})$ in $\mathbf{Br}_h(K)$ to define a ring structure. The associativity property will then require that reducing $(A^{\otimes 3}, \sigma^{\otimes 3})$ to (A, σ) does not depend on which factor we apply the reduction to. In general, we want a well-defined map from $(A^{\otimes d}, \sigma^{\otimes d})$ to (A, σ) if d is odd, and to (K, Id) if d is even. We also want all this to be functorial in $\mathbf{Br}_h(K)$. This is achieved by the formalism we describe in this section (see proposition 1.28).

If $(C, \otimes, I)$ is a symmetric monoidal category, then for any $n \geqslant 2$ we write

$$C[n] = \mathrm{Hom}^s_\otimes(\langle \mathbb{Z}/n\mathbb{Z}\rangle, C),$$

and we call it the n-torsion category of C. There is a natural functor

$$C[n] \longrightarrow C$$

sending F to $F(1)$. Of course this is inspired by the inclusion morphism $M[n] \to M$ when M is a commutative monoid, and the isomorphism $M[n] \simeq \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, M)$.

Definition 1.26. *Let $(C, \otimes, I)$ be a symmetric monoidal category, and let $n \geqslant 2$. We say that C is a strong symmetric n-torsion 2-group if C is a groupoid such that the canonical functor $C[n] \to C$ is an equivalence of categories.*

Example 1.27. If M is a commutative monoid, then $\langle M \rangle$ is a strong symmetric n-torsion 2-group iff M is a n-torsion group.

Proposition 1.28. *Let $(C, \otimes, I)$ be a strong symmetric n-torsion 2-group for some $n \geqslant 2$. Then C is strongly symmetric, and for any object $x \in C$ and any $d \in \mathbb{N}$, there is an isomorphism*

$$\varphi_x^{(d)} : x^{\otimes d} \longrightarrow x^{\otimes r} \tag{6}$$

with $0 \leqslant r < n$ and $d \equiv r$ modulo n, which satisfies that

$$\begin{array}{ccc} x^{\otimes d_1 + d_2} & & \\ {\scriptstyle \varphi_x^{(d_1)} \otimes \varphi_x^{(d_2)}} \downarrow & \searrow^{\varphi_x^{(d_1+d_2)}} & \\ x^{\otimes r_1 + r_2} & \xrightarrow[\varphi_x^{(r_1+r_2)}]{} & x^{\otimes r} \end{array} \tag{7}$$

commutes, as well as

$$\begin{array}{ccc} x^{\otimes d} & \xrightarrow{f^{\otimes d}} & y^{\otimes d} \\ {\scriptstyle \varphi_x^{(d)}} \downarrow & & \downarrow {\scriptstyle \varphi_y^{(d)}} \\ x^{\otimes r} & \xrightarrow{f^{\otimes r}} & y^{\otimes r} \end{array} \tag{8}$$

for any morphism $f : x \to y$ in C.

Proof. Let $\Phi : C \to C[n]$ be an inverse of the canonical $C \to C[n]$. Then up to a natural monoidal isomorphism we may assume that $\Phi(x)$ satisfies $\Phi(x)(1) = x$, and since $\Phi(x)$ is monoidal we may even assume that $\Phi(x)(r) = x^{\otimes r}$ for any $0 \leqslant r < n$. The fact that $\Phi(x)$ is symmetric shows that

$$\begin{array}{ccc} \Phi(x)(1) \otimes \Phi(x)(1) = x \otimes x & \xrightarrow{\quad s \quad} & x \otimes x = \Phi(x)(1) \otimes \Phi(x)(1) \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle f} & \\ & \Phi(x)(2) & \end{array}$$

commutes, where s is the switch morphism and f is an isomorphism given by the monoidal structure of $\Phi(x)$, so s is the identity and C is strongly symmetric.

The fact that $\Phi(x)$ is monoidal also yields a natural isomorphism

$$\Phi(x)(1)^{\otimes d} \longrightarrow \Phi(x)(r)$$

for any $d \in \mathbb{N}$, where r is the residue of d modulo n, which defines $\varphi_x^{(d)}$. Then the fact that (7) commutes is a consequence of the fact that $\Phi(x)$ is monoidal, and for (8), it follows from the fact that Φ is a functor. □

Now to establish strong n-torsion, we may use:

Lemma 1.29. *Let $(C, \otimes, I)$ be a strongly symmetric monoidal groupoid, and let $n \geqslant 2$. Suppose that for any object $x \in C$ we can choose a morphism $\varphi_x : x^{\otimes n} \to I$ such that for all morphisms $f : x \to y$ the following diagram commutes:*

$$\begin{array}{ccc} x^{\otimes n} & \xrightarrow{\quad f^{\otimes n} \quad} & y^{\otimes n} \\ & {\scriptstyle \varphi_x} \searrow \quad \swarrow {\scriptstyle \varphi_y} & \\ & I. & \end{array} \tag{9}$$

Then C is a strong symmetric n-torsion 2-group, and we can take $\varphi_x^{(n)} = \varphi_x$ in proposition 1.28.

Proof. Consider the functor $\Phi(x) : \langle \mathbb{Z}/n\mathbb{Z} \rangle \to C$ given by $r \mapsto x^{\otimes r}$ for $0 \leqslant r < n$. Then to define a monoidal structure on $\Phi(x)$ we have to give morphisms $x^{\otimes a+b} \to x^{\otimes r}$ where $0 \leqslant a, b < n$ and $a + b = qn + r$ is the euclidean division of $a + b$ by n. If $q = 0$, we take the identity, and if $q = 1$ we take $\varphi_x \otimes 1_{x^{\otimes r}}$. It is not difficult to check that this defines a monoidal functor structure if $\varphi_x \otimes 1_x : x^{\otimes n+1} \to x$ and $1_x \otimes \varphi_x : x^{\otimes n+1} \to x$ coincide. But they coincide up to the action of a $(n+1)$-cycle on $x^{\otimes n+1}$ (using the symmetric structure of C), and since C is strongly symmetric this is the identity. So $\Phi(x)$ is a monoidal functor, and since C is strongly symmetric it is actually a symmetric functor, which means that $\Phi(x) \in C[n]$.

Now we have to verify that if $f : x \to y$ is a morphism in C, the induced natural transformation $\Phi(x) \to \Phi(y)$ is a morphism in $C[n]$, which is to say that it is a monoidal transformation. Given the monoidal structures on $\Phi(x)$ and $\Phi(y)$, this is precisely (9). □

Then we can state:

Theorem 1.30. *The symmetric monoidal category $\mathbf{Br}_h(K)$ is a strong symmetric 2-torsion 2-group, with the structure described in proposition 1.28 given by*

$$\varphi^{(2)}_{(A,\sigma)} : (A \otimes_K A, \sigma \otimes \sigma) \xrightarrow{(|A|_\sigma, T_\sigma)} (K, \mathrm{Id})$$

(see example 1.9).

Proof. According to lemma 1.29, we have to show that for any $f : (B,\tau) \to (A,\sigma)$ in $\mathbf{Br}_h(K)$, the diagram

$$\begin{array}{ccc} (B \otimes_K B, \tau \otimes \tau) & \xrightarrow{f^{\otimes 2}} & (A \otimes_K A, \sigma \otimes \sigma) \\ & {\scriptstyle (|B|_\tau, T_\tau)} \searrow \quad \swarrow {\scriptstyle (|A|_\sigma, T_\sigma)} & \\ & I & \end{array}$$

commutes. If f corresponds to the ε-hermitian bimodule (V,h), then we define the following $(B \otimes_K B)$-K-bimodule morphism:

$$\begin{array}{rrcl} \psi : & (V \otimes_K V) \otimes_{A \otimes_K A} |A|_\sigma & \longrightarrow & |B|_\tau \\ & (v \otimes w) \otimes a & \longmapsto & \varphi_h(va \otimes w). \end{array}$$

where $\varphi_h : V \otimes_K V \to B$ (see [5, 5.1]) is characterized by

$$\forall x \in V,\ \varphi_h(v \otimes w) \cdot x = vh(w,x).$$

Then ψ is well-defined since for $x, y \in A$:

$$\begin{aligned} \psi((v \otimes w) \otimes (xa\sigma(y)) &= \varphi_h(vxa\sigma(y) \otimes w) \\ &= \varphi_h(vxa \otimes wy) \\ &= \psi((vx \otimes wy) \otimes a), \end{aligned}$$

and it is a bimodule morphism since for $x, y \in B$:

$$\begin{aligned} \psi((xv \otimes yw) \otimes a) &= \varphi_h(xva \otimes yw) \\ &= x\varphi_h(va \otimes v)\tau(y). \end{aligned}$$

To show that ψ is an isometry, we must establish equality between on the one hand

$$\begin{aligned} &\mathrm{Trd}_B\left(\tau\left(\psi((v \otimes w) \otimes a)\right) \cdot \psi(((v' \otimes w') \otimes b))\right) \\ &= \mathrm{Trd}_B\left(\tau(\varphi_h(va \otimes w)) \cdot \varphi_h(v'b \otimes w')\right) \end{aligned}$$

and on the other hand

$$\begin{aligned} &\mathrm{Trd}_A\left(\sigma(a)(h \otimes h)(v \otimes w, v' \otimes w') \cdot b\right) \\ &= \varepsilon\,\mathrm{Trd}_A(\sigma(a)h(v,v')bh(w',w)). \end{aligned}$$

Now applying successively the formulas in theorem [5, 5.1], we get:

$$\begin{aligned} &\mathrm{Trd}_B\left(\tau(\varphi_h(va \otimes w)) \cdot \varphi_h(v'b \otimes w')\right) \\ &= \varepsilon\,\mathrm{Trd}_B\left(\varphi_h(w \otimes va) \cdot \varphi_h(v'b \otimes w')\right) \\ &= \varepsilon\,\mathrm{Trd}_B\left(\varphi_h(wh(va,v'b) \otimes w')\right) \\ &= \varepsilon\,\mathrm{Trd}_A\left(h(w', wh(va,v'b))\right) \\ &= \varepsilon\,\mathrm{Trd}_A\left(h(w',w)\sigma(a)h(v,v')b\right). \end{aligned}$$

□

2 The mixed Witt ring of an algebra with involution

We now want to use the hermitian Morita theory developped in the first section to define a product such that $\widetilde{GW}(A,\sigma)$ and $\widetilde{W}(A,\sigma)$ are commutative graded rings, as described in the introduction.

2.1 The mixed Witt group

We start by defining the group structure for the various rings.

Definition 2.1. *Let (A,σ) be an algebra with involution over K, and let $\varepsilon = \pm 1$. We denote by $SW^\varepsilon(A,\sigma)$ (resp. $SW_\varepsilon(A,\sigma)$) the set of isometry classes of hermitian modules over (A,σ) with sign (resp. type) ε (see definition 1.10).*

It is a semi-group when equipped with the orthogonal direct sum of ε-hermitian modules. We set

$$SW^\pm(A,\sigma) = SW^1(A,\sigma) \oplus SW^{-1}(A,\sigma) = SW_1(A,\sigma) \oplus SW_{-1}(A,\sigma).$$

We will often write $SW(A,\sigma)$ for $SW^1(A,\sigma)$, and if $(A,\sigma) = (K,\mathrm{Id})$ we also write $SW^\varepsilon(K)$, $SW_\varepsilon(K)$ and $SW^\pm(K)$.

Then we define the mixed Witt semi-group of (A,σ) as

$$\widetilde{SW}(A,\sigma) = SW^\pm(K) \oplus SW^\pm(A,\sigma).$$

Remark 2.2. The set $SW^\varepsilon(A,\sigma)$ (resp. $SW_\varepsilon(A,\sigma)$) is the set of morphisms in $\mathbf{Br}_h(K)$ with target (A,σ) and sign (resp. type) ε, modulo the action of algebraic isomorphisms on the source objects (see remark 1.18).

Remark 2.3. By definition, $SW_\varepsilon(A,\sigma) = SW^{\varepsilon_\sigma \varepsilon}(A,\sigma)$, where $\varepsilon_\sigma = 1$ if σ is orthogonal, and $\varepsilon_\sigma = -1$ if σ is symplectic. In particular, $SW^\varepsilon(K) = SW_\varepsilon(K)$.

The semi-group $\widetilde{SW}(A,\sigma)$ is $\mathbb{Z}/2\mathbb{Z}$-graded, where we call $SW^\pm(K)$ the *even* component, and $SW^\pm(A,\sigma)$ the *odd* component.

There is also a functoriality with respect to $\mathbf{Br}_h(K)$: if $f : (B,\tau) \to (A,\sigma)$ is a morphism in $\mathbf{Br}_h(K)$, then we can define an application

$$f_* : \widetilde{SW}(B,\tau) \longrightarrow \widetilde{SW}(A,\sigma)$$

which is the identity on the even component, and acts on $SW^\varepsilon(B,\tau)$ as the composition by f on the right; an element x of $SW^\varepsilon(B,\tau)$ corresponds to some morphism $g : (C,\theta) \to (B,\tau)$ modulo an algebraic automorphism of (C,θ) (see remark 2.2), and we define $f_*(x) = f \circ g$, which is also well-defined up to an algebraic automorphism of (C,θ). This defines a map $SW^\varepsilon(B,\tau) \to SW^{\varepsilon\varepsilon_f}(A,\sigma)$, and thus a map $SW^\pm(B,\tau) \to SW^\pm(A,\sigma)$.

Note that if we consider the *type* of hermitian modules instead of the *sign*, we get a better behaviour, according to remark 1.15: f_* is a map $SW_\varepsilon(B,\tau) \to SW_\varepsilon(A,\sigma)$. This leads us to define a certain grading on $\widetilde{SW}(A,\sigma)$:

Definition 2.4. *We define the abelian group $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mu_2(K)$. Let (A,σ) be an algebra with involution over K. We endow $\widetilde{SW}(A,\sigma)$ with a Γ-grading such that the component of degree $([i],\varepsilon) \in \Gamma$ with $i \in \{0,1\}$ is $SW_\varepsilon((A,\sigma)^{\otimes i})$.*

Proposition 2.5. *The association $(A,\sigma) \mapsto \widetilde{SW}(A,\sigma)$ and $f \mapsto f_*$ defines a functor from $\mathbf{Br}_h(K)$ to the category of commutative Γ-graded semi-groups.*

Proof. The fact that this is a functor to the category of sets is clear by associativity of the composition in $\mathbf{Br}_h(K)$. We have discussed above that the Γ-grading was precisely chosen to be preserved by the maps f_*. It remains to show that the f_* preserve the sum, but this is clear since the composition with f is given by a tensor product, which is distributive over direct sums. □

Remark 2.6. The semi-groups $\widetilde{SW}(A,\sigma)$ are equipped with a distinguished element, namely $\langle 1\rangle_\sigma \in SW(A,\sigma)$, and through functoriality those distinguished elements are in some sense generic. Precisely, if $x \in SW^\varepsilon(A,\sigma)$, then x corresponds to some $f : (B,\tau) \to (A,\sigma)$, and then $x = f_*(\langle 1\rangle_\tau)$.

We can now use those semi-groups to define the groups we are interested in. Recall that a ε-hermitian module (V,h) is hyperbolic when there exists a submodule $W \subset V$ such that $W = W^\perp$ (such a W is called a Lagrangian for (V,h)).

Lemma 2.7. *Let (A,σ) and (B,τ) be algebras with involution over K, and let (V,h) be a hyperbolic ε-hermitian module over (B,τ).*

If $f : (B,\tau) \to (A,\sigma)$ is a morphism in $\mathbf{Br}_h(K)$, then $f_*(V,h)$ is hyperbolic over (A,σ).

If (U,h') is a ε'-hermitian module over (A,σ), then $(U \otimes_K V, h' \otimes h)$ is hyperbolic over $(A \otimes_K B, \sigma \otimes \tau)$.

Proof. It is easy to see that if $W \subset V$ is a Lagrangian, then $f_*(W, h_{|W})$ is a Lagrangian for $f_*(V,h)$, and $U \otimes_K W$ is a Lagrangian for $U \otimes_K V$. □

Definition 2.8. *Let (A,σ) be an algebra with involution over K, and let $\varepsilon = \pm 1$. Then $GW^\varepsilon(A,\sigma)$ (resp. $GW_\varepsilon(A,\sigma)$) is the Grothendieck group of $SW^\varepsilon(A,\sigma)$ (resp. $SW_\varepsilon(A,\sigma)$), and $W^\varepsilon(A,\sigma)$ (resp. $W_\varepsilon(A,\sigma)$) is its quotient by the subgroup of hyperbolic modules.*

Furthermore, the mixed Grothendieck-Witt group $\widetilde{GW}(A,\sigma)$ is the Grothendieck group of $\widetilde{SW}(A,\sigma)$, and the mixed Witt group $\widetilde{W}(A,\sigma)$ is the quotient by the (homogeneous) subgroup generated by hyperbolic modules. They are Γ-graded groups with

$$\widetilde{GW}(A,\sigma) = GW^\pm(K) \oplus GW^\pm(A,\sigma)$$
$$\widetilde{W}(A,\sigma) = W(K) \oplus W^\pm(A,\sigma).$$

Remark 2.9. Note that the component of degree $(0,-1) \in \Gamma$ of $\widetilde{W}(A,\sigma)$ is trivial, since all anti-symmetric bilinear forms are hyperbolic.

We easily deduce from proposition 2.5:

Proposition 2.10. *The functor $\widetilde{SW}$ induces functors $\widetilde{GW}$ and $\widetilde{W}$ from $\mathbf{Br}_h(K)$ to the category of commutative Γ-graded groups.*

Proof. For $\widetilde{GW}$, this is a direct consequence of proposition 2.5, since by universal property, a (graded) semi-group morphism extends uniquely to a (graded) group morphism between the Grothendieck groups. For $\widetilde{W}$, we use lemma 2.7 to see that f_* sends hyperbolic modules to hyperbolic modules. □

2.2 The mixed Witt ring

Now we define the ring structures, in order to get Γ-graded rings. The key point is theorem 1.30, through the use of the morphisms $\varphi^{(d)}_{(A,\sigma)}$ in $\mathbf{Br}_h(K)$ (see (6)).

Definition 2.11. *Let (A,σ) be an algebra with involution, and let $g_1 = (i_1, \varepsilon_1)$ and $g_2 = (i_2, \varepsilon_2)$ be two elements of Γ, with $i_k \in \{0,1\}$. We define the product of homogeneous elements of $\widetilde{SW}(A,\sigma)$ with respective degrees g_1 and g_2 by:*

$$\begin{array}{ccc} SW_{\varepsilon_1}((A,\sigma)^{\otimes i_1}) \times SW_{\varepsilon_2}((A,\sigma)^{\otimes i_2}) & \longrightarrow & SW_{\varepsilon_1\varepsilon_2}((A,\sigma)^{\otimes i_1+i_2}) \\ & \searrow & \downarrow (\varphi^{(i_1+i_2)}_{(A,\sigma)})_* \\ & & SW_{\varepsilon_1\varepsilon_2}((A,\sigma)^{\otimes i}) \end{array}$$

where $i \in \{0,1\}$ is such that $i_1 + i_2 \equiv i$ modulo 2.
We naturally extend this bilinearly to a binary product on $\widetilde{SW}(A,\sigma)$.

Note that since $\varphi^{(d)}_{(A,\sigma)}$ is the identity when $d = 0, 1$, if i_1 or i_2 is 0 then we retrieve the usual product of a hermitian form over (A,σ) and a bilinear form over K, given by the tensor product. In particular, this gives the usual semi-ring structure on $SW^{\pm}(K)$.

The new ingredient is the product

$$SW_{\varepsilon_1}(A,\sigma) \times SW_{\varepsilon_2}(A,\sigma) \longrightarrow SW_{\varepsilon_1\varepsilon_2}(K)$$

which is by definition

$$(V_1 \cdot V_2, h_1 \cdot h_2) := (V_1, h_1) \cdot (V_2, h_2) = (|A|_\sigma, T_\sigma) \circ (V_1 \otimes_K V_2, h_1 \otimes h_2).$$

By definition of the composition in $\mathbf{Br}_h(K)$ (see 1.11), we have

$$V_1 \cdot V_2 = (V_1 \otimes_K V_2) \otimes_{A \otimes_K A} |A|_\sigma, \tag{10}$$

and using (4):

$$(h_1 \cdot h_2)(u_1 \otimes u_2 \otimes a, v_1 \otimes v_2 \otimes b) = \mathrm{Trd}_A\left(\sigma(a) h_1(u_1, v_1) b \sigma(h_2(u_2, v_2))\right). \tag{11}$$

Proposition 2.12. *This products defines a Γ-graded commutative semi-ring structure on $\widetilde{SW}(A,\sigma)$, which induces naturally a Γ-graded commutative ring structure on $\widetilde{GW}(A,\sigma)$. Furthermore, the subgroup of $\widetilde{GW}(A,\sigma)$ generated by hyperbolic modules is a homogeneous ideal, so $\widetilde{W}(A,\sigma)$ is also naturally a Γ-graded commutative ring.*

Proof. The fact that the product on $\widetilde{SW}(A,\sigma)$ is distributive follows from the fact that the tensor product is distributive over direct sums, and composition with $\varphi^{(d)}_{(A,\sigma)}$ preserves direct sums (since it is also a tensor product).

For commutativity, if $f : (B,\tau) \to (A,\sigma)^{\otimes i}$ and $g : (C,\theta) \to (A,\sigma)^{\otimes j}$ are two morphisms in $\mathbf{Br}_h(K)$, with $i, j \in \{0,1\}$, then since $\mathbf{Br}_h(K)$ is strongly symmetric, the natural diagram

$$\begin{array}{ccc} (B \otimes_K C, \tau \otimes \theta) & \xrightarrow{\Theta(s)} & (C \otimes_K B, \theta \otimes \tau) \\ & {\scriptstyle f \otimes g} \searrow \quad \swarrow {\scriptstyle g \otimes f} & \\ & (A,\sigma)^{\otimes i+j} & \end{array}$$

commutes, where s is the switch map in $\mathbf{AlgInv}(K)$. Thus $\varphi^{(i+j)}_{(A,\sigma)} \circ (f \otimes g)$ and $\varphi^{(i+j)}_{(A,\sigma)} \circ (g \otimes f)$ differ by an algebraic isomorphism, and they define the same element in $\widetilde{SW}(A,\sigma)$.

For associativity, if for $1 \leqslant k \leqslant 3$ we are given $f_k : (B_k, \tau_k) \to (A,\sigma)^{\otimes i_k}$, morphism in $\mathbf{Br}_h(K)$ (with $i_k \in \{0,1\}$), then since the tensor product is associative we have $f_1 \otimes (f_2 \otimes f_3) = (f_1 \otimes f_2) \otimes f_3$ as morphisms to $(A,\sigma)^{\otimes i_1+i_2+i_3}$, and we can conclude since the following diagram commutes, as can be seen by applying (7):

$$\begin{array}{ccc} (A,\sigma)^{\otimes i_1+i_2+i_3} & \xrightarrow{Id_{(A,\sigma)}\otimes\varphi^{(i_2+i_3)}_{(A,\sigma)}} & (A,\sigma)^{\otimes i_1+r_2} \\ {\scriptstyle \varphi^{(i_1+i_2)}_{(A,\sigma)}\otimes Id_{(A,\sigma)}}\downarrow & & \downarrow{\scriptstyle \varphi^{(i_1+r_2)}_{(A,\sigma)}} \\ (A,\sigma)^{\otimes r_1+i_3} & \xrightarrow{\varphi^{(r_1+i_3)}_{(A,\sigma)}} & (A,\sigma)^{\otimes r}. \end{array}$$

Thus $\widetilde{SW}(A,\sigma)$ is indeed a commutative Γ-graded semi-ring. Now by universal property of Grothendieck groups this extends uniquely to a structure of Γ-graded commutative ring on $\widetilde{GW}(A,\sigma)$. It just remains to show that the group generated by hyperbolic modules is an ideal of $\widetilde{GW}(A,\sigma)$. This follows from lemma 2.7, since is shows that a tensor product of a hermitian module with a hyperbolic module is hyperbolic, and composing with $\varphi^{(d)}_{(A,\sigma)}$ yields a hyperbolic module. □

Hence, we will respectively call $\widetilde{SW}(A,\sigma)$, $\widetilde{GW}(A,\sigma)$ and $\widetilde{W}(A,\sigma)$ the mixed Witt semi-ring, Grothendieck-Witt ring and Witt ring of (A,σ).

Remark 2.13. In the case of a quaternion algebra Q with its canonical involution γ, Lewis makes in [7] a very similar construction to our $\widetilde{W}(Q,\gamma)$. His definition amounts to essentially the same, except that he uses the norm form of Q instead of the involution trace form T_γ. Since these two forms differ by a factor $\langle 2\rangle$, we get non-isomorphic but very similar rings. However, the norm form is a special feature of quaternion algebras (in general for an algebra of degree n the reduced norm is a homogeneous polynomial function of degree n), so the construction does not generalize well to arbitrary algebras. Furthermore, no proof of the associativity of the product is given in [7].

Example 2.14. We have by construction $\widetilde{GW}(K,\mathrm{Id}) = GW^{\pm}(K) \oplus GW^{\pm}(K)$ and $\widetilde{W}(K,\mathrm{Id}) = W(K) \oplus W(K)$ as Γ-graded groups. It is easy to see by definition of the product that as Γ-graded rings, we have canonical isomorphisms $\widetilde{GW}(K,\mathrm{Id}) \simeq GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$ and $\widetilde{W}(K,\mathrm{Id}) \simeq W(K)[\mathbb{Z}/2\mathbb{Z}]$ (where $GW^{\pm}(K)$ and $W(K)$ are considered as μ_2-graded rings).

Example 2.15. In particular, if (A,σ) is a split algebra with orthogonal involution, then there is a ring isomorphism $\widetilde{W}(A,\sigma) \approx W(K)[\mathbb{Z}/2\mathbb{Z}]$, but it depends on the choice of a quadratic form q such that $\sigma = \sigma_q$ (which is only well-defined up to a scalar factor).

We can also extend the functoriality to these ring structures:

Theorem 2.16. *The constructions $\widetilde{SW}$, $\widetilde{GW}$ and $\widetilde{W}$ define functors from $\mathbf{Br}_h(K)$ to the respective categories of Γ-graded commutative semi-rings, Γ-graded commutative $GW^{\pm}(K)$-algebras and Γ-graded commutative $W(K)$-algebras.*

Proof. If $f : (B, \tau) \to (A, \sigma)$ is a morphism in $\mathbf{Br}_h(K)$, then we want to show that $f_* : \widetilde{SW}(B, \tau) \to \widetilde{SW}(A, \sigma)$ is actually a semi-ring morphism. This follows by definition from the fact that

$$\begin{array}{ccc} (B,\tau)^{\otimes i_1+i_2} & \xrightarrow{f^{\otimes i_1+i_2}} & (A,\sigma)^{\otimes i_1+i_2} \\ {\scriptstyle \varphi^{(i_1+i_2)}_{(B,\tau)}}\downarrow & & \downarrow{\scriptstyle \varphi^{(i_1+i_2)}_{(A,\sigma)}} \\ (B,\tau)^{\otimes i} & \xrightarrow{f^{\otimes i}} & (A,\sigma)^{\otimes i} \end{array}$$

commutes, which is a special case of (8).

Then by universal property of Grothendieck rings of semi-rings, f_* extends uniquely to a ring morphism from $\widetilde{GW}(B, \tau)$ to $\widetilde{GW}(A, \sigma)$, and since f_* preserves hyperbolic modules (see lemma 2.7), it also induces a ring morphism from $\widetilde{W}(B, \tau)$ to $\widetilde{W}(A, \sigma)$. □

Example 2.17. Let us consider the case where $f = \langle a \rangle_\sigma : (A, \sigma_a) \to (A, \sigma)$. Let (V_i, h_i) be a ε_i-hermitian module over (A, σ_a) for $i = 1, 2$, with $B_i = \mathrm{End}_A(V_i)$ and $\tau_i = \sigma_{h_i}$. We should have an isometry between the bilinear spaces $f_*((V_1, h_1) \cdot (V_2, h_2))$, which is just $(V_1, h_1) \cdot (V_2, h_2)$, and $f_*(V_1, h_1) \cdot f_*(V_2, h_2)$.

Now according to example 1.12 we see that $f_*(V_i, h_i)$ has underlying bimodule V_i, with the form $h'_i(x, y) = h_i(xa, y)$. Thus $(V_1, h_1) \cdot (V_2, h_2)$ is a bilinear space with underlying space $(V_1 \otimes_K V_2) \otimes_{A \otimes_K A} |A|_{\sigma_a}$, while $f_*(V_1, h_1) \cdot f_*(V_2, h_2)$ has underlying space $(V_1 \otimes_K V_2) \otimes_{A \otimes_K A} |A|_\sigma$. The isometry between the two is induced by the bimodule isomorphism

$$\begin{array}{ccc} |A|_\sigma & \longrightarrow & |A|_{\sigma_a} \\ x & \longmapsto & xa. \end{array}$$

Example 2.18. Since $\widetilde{GW}$ (resp. $\widetilde{W}$) is a functor, any automorphism of (A, σ) in $\mathbf{Br}_h(K)$ induces an automorphism of $\widetilde{GW}(A, \sigma)$ (resp. $\widetilde{W}(A, \sigma)$); we call such an automorphism *standard*. Recall from proposition 1.17 that the automorphisms of (A, σ) are the $\langle \lambda \rangle_\sigma$ where $\lambda \in K^*$. The associated standard automorphism is the identity on $GW^{\pm}(K)$ (resp. $W(K)$), and is the multiplication by $\langle \lambda \rangle$ on $GW^{\pm}(A, \sigma)$ (resp. $W^{\pm}(A, \sigma)$). It is easy to check directly that this is indeed an automorphism.

Remark 2.19. Since $\mathbf{Br}_h(K)$ is a groupoid, this means that all induced maps f_* are isomorphisms. Now seeing that (A, σ) and (B, τ) are isomorphic in $\mathbf{Br}_h(K)$ iff A and B are Brauer-equivalent, we can conclude that $\widetilde{GW}(A, \sigma)$ and $\widetilde{W}(A, \sigma)$ only depend on the Brauer class of A, up to a graded ring isomorphism. Thus we may if necessary reduce to the study of division algebras with involution: if D is the division algebra equivalent to A, and if θ is any involution on D, then $\widetilde{GW}(A, \sigma) \approx \widetilde{GW}(D, \theta)$ and $\widetilde{W}(A, \sigma) \approx \widetilde{W}(D, \theta)$; but note that the isomorphism is non-canonical. Precisely, it is determined up to a standard automorphism.

2.3 Twisted involution trace forms

Now that we have established the formal properties of our rings, we would like to be able to perform explicit computations. Obviously we can compute any product if we can describe products of homogeneous elements in $\widetilde{SW}(A, \sigma)$, and a quick examination of the cases shows that only products $xy \in SW(K)$ with $x, y \in SW^\varepsilon(A, \sigma)$ present a difficulty. It is natural to specialize to the case where x and y are diagonal, and by bilinearity we just need to describe products of the form $\langle a\rangle_\sigma \cdot \langle b\rangle_\sigma$.

Proposition 2.20. *Let (A, σ) be an algebra with involution over K, and let $a, b \in \mathrm{Sym}^\varepsilon(A^\times, \sigma)$. Then in $\widetilde{SW}(A, \sigma)$ we have*

$$\langle a\rangle_\sigma \cdot \langle b\rangle_\sigma = T_{\sigma,a,b}$$

where $T_{\sigma,a,b}$ is the symmetric bilinear form given by

$$\begin{array}{rcl} |A|_\sigma \times |A|_\sigma & \longrightarrow & K \\ (x, y) & \longmapsto & \mathrm{Trd}_A(\sigma(x)ay\sigma(b)). \end{array}$$

Proof. By definition (and example 1.6), $\langle a\rangle_\sigma \cdot \langle b\rangle_\sigma$ is the bilinear form given by the composition $(|A|_\sigma, T_\sigma) \circ (|A \otimes_K A|, \langle a \otimes b\rangle_{\sigma\otimes\sigma})$. According to example 1.12, this is

$$(x, y) \mapsto T_\sigma(x, (a \otimes b) \cdot y).$$

Since $T_\sigma(x, y) = \mathrm{Trd}_A(\sigma(x)y)$ and $(a \otimes b) \cdot y = ay\sigma(b)$, we may conclude. □

Example 2.21. In particular, $\langle 1\rangle_\sigma^2 = T_\sigma$, which of course follows directly from the definition of the product. The idea that T_σ represents in some sense the "square" of the involution σ has appeared in the literature in various forms, for instance in the definition of the signature of an involution ([5, 11.10]). Our construction gives some solid ground to this idea.

Example 2.22. More generally, $\langle a\rangle_\sigma \cdot \langle 1\rangle_\sigma = T_{\sigma,a}$, where $T_{\sigma,a} = T_{\sigma,a,1}$ is called a twisted involution trace form in [5, §11].

Remark 2.23. We can entirely characterize the ring structure of $\widetilde{GW}(A, \sigma)$ and $\widetilde{W}(A, \sigma)$ by theorem 2.16 and proposition 2.20. Indeed, we can use them to compute any product of ε-hermitian forms: if $h, h' \in SW^\varepsilon(A, \sigma)$, choose any involution θ on the division algebra D in the Brauer class of A, and choose some morphism $f : (A, \sigma) \to (D, \theta)$ in $\mathbf{Br}_h(K)$ (which amounts to choosing a hermitian form over (D, θ) whose adjoint involution is σ). Then $f_*(h) = \langle a_1, \dots, a_n\rangle_\theta$ and $f_*(h') = \langle b_1, \dots, b_m\rangle_\theta$, so $h \cdot h' = \sum_{i,j} T_{\theta,a_i,b_j}$. Note that it is not obvious at first sight that this quadratic form is independant of the choice of diagonalizations.

Products of the form $\langle a\rangle_\sigma \cdot \langle b\rangle_\sigma$ are special cases of products $h \cdot h'$ where h and h' are defined on the same module. We can actually treat all such products similarly:

Corollary 2.24. *Let (A, σ) be an algebra with involution over K, V be a right A-module, and h, h' be two ε-hermitian forms on V. If we set $B = \mathrm{End}_A(V)$ and $\tau = \sigma_h$, then there is a unique $u \in B^\times$ such that for all $x, y \in V$, $h'(x, y) = h(ux, y)$. Furthermore, $\tau(u) = u$, $\sigma_{h'} = \tau_u$, and*

$$h' \cdot h = T_{\tau,u}$$

as a product in $\widetilde{GW}(A, \sigma)$. In particular, $h^2 = T_\tau$.

Proof. For a fixed x, $h'(x,-)$ is a A-linear map $V \to A$, so since h is regular there exists a unique x' such that $h'(x,-) = h(x',-)$. It is easy to see that $x \mapsto x'$ is A-linear, which shows existence and uniqueness of u (which is invertible since h' is also regular). We easily see that $\tau(u) = u$ using that h' is ε-hermitian.

Let $f : (B,\tau) \to (A,\sigma)$ be the morphism in $\mathbf{Br}_h(K)$ corresponding to (V,h). Then $f_*(\langle 1\rangle_\tau) = (V,h)$, and $f_*(\langle u\rangle_\tau) = (V,h')$ (see example 1.12). Thus since f_* is a ring morphism we find $h' \cdot h = \langle u\rangle_\tau \cdot \langle 1\rangle_\tau = T_{\tau,u}$. □

This means that we can reinterpret twisted involution trace forms (as in [5, §11]) as being exactly the products of ε-hermitian forms defined on the same module. These computations show that understanding the product in $\widetilde{GW}(A,\sigma)$ and $\widetilde{W}(A,\sigma)$ amounts to understanding twisted involution trace forms (usually for involutions different from σ). From the associativity of $\widetilde{GW}(A,\sigma)$ we can deduce:

Corollary 2.25. *Let (A,σ) and (B,τ) be algebras with involution over K, with A and B Brauer-equivalent, and let h be a ε-hermitian form over (A,σ). If $\varepsilon = -1$ we assume that σ and τ have the same type, and if $\varepsilon = 1$ we assume they have different type. Then $T_\tau \cdot h \in GW^\varepsilon(A,\sigma)$ is hyperbolic.*

Proof. Let $h' \in GW^{-\varepsilon}(A,\sigma)$ such that $\sigma_{h'} = \tau$. Then $T_\tau \cdot h = (h' \cdot h') \cdot h$ in $\widetilde{GW}(A,\sigma)$. But $h' \cdot h \in GW^{-1}(K)$, so it is hyperbolic. □

We can compute the products in mixed Witt rings of quaternion algebras, imitating the proof of [5, 11.6]. Recall that if Q is a quaternion algebra, then its reduced norm map is a quadratic form on Q, denoted $n_Q \in GW(K)$, and it is the unique 2-fold Pfister form whose Clifford invariant $e_2(n_Q) \in H^2(K,\mu_2)$ is the Brauer class of Q.

For any pure quaternions $z_1, z_2 \in Q$, the Brauer class $[Q]$ and the symbol $(z_1^2, z_2^2) \in H^2(K,\mu_2)$ have a common slot (for instance z_1^2), so $[Q] + (z_1^2, z_2^2)$ is a symbol. We write φ_{z_1,z_2} for the unique 2-fold Pfister form whose Clifford invariant is this symbol. In particular, if z_1 and z_2 anti-commute, φ_{z_1,z_2} is hyperbolic.

Proposition 2.26. *Let (Q,γ) be a quaternion algebra over K endowed with its canonical symplectic involution. Then for any $a, b \in K^*$ we have in $\widetilde{W}(Q,\gamma)$:*

$$\langle a\rangle_\gamma \cdot \langle b\rangle_\gamma = \langle 2ab\rangle n_Q \in W(K).$$

Furthermore, for any pure quaternions $z_1, z_2 \in Q^\times$, we have:

$$\langle z_1\rangle_\gamma \cdot \langle z_2\rangle_\gamma = \begin{cases} 0 \in W(K) & \text{if } z_1 \text{ and } z_2 \text{ anti-commute} \\ \langle -\operatorname{Trd}_Q(z_1 z_2)\rangle \varphi_{z_1,z_2} \in W(K) & \text{otherwise.} \end{cases}$$

Proof. From proposition 2.20 we see that

$$\langle a\rangle_\gamma \cdot \langle b\rangle_\gamma = \langle ab\rangle T_\gamma,$$

and it is easy to see that $T_\gamma = \langle 2\rangle n_Q$ (for instance by taking a standard quaternionic basis of Q). For the second formula, we can check that if z_1 and z_2 anti-commute then $K \oplus K z_1$ is a Lagrangian in $(|Q|, T_{\gamma,z_1,z_2})$, and otherwise we can check that if $z \in Q^\times$ anti-commutes with z_1, then the basis $(1, z_1, zz_2, z_1zz_2)$ is orthogonal for T_{γ,z_1,z_2}, giving the diagonalization

$$\langle -\operatorname{Trd}_Q(z_1z_2)\rangle\langle 1, -z_1^2, -z_2^2z^2, z_1^2z_2^2z^2\rangle$$

which shows that $T_{\gamma,z_1,z_2} \simeq \langle -\operatorname{Trd}_Q(z_1z_2)\rangle\langle\langle z_1^2, z_2^2z^2\rangle\rangle$, and it is easy to see that by definition $\varphi_{z_1,z_2} = \langle\langle z_1^2, z_2^2z^2\rangle\rangle$, since $[Q] = (z_1^2, z^2)$. ☐

2.4 Scalar extension and reciprocity

Scalar extension is a standard tool in the theory of algebras with involution, in particular when extending the scalars to a splitting field to reduce to the classical theory of bilinear forms over fields. Recall from (5) the canonical functor $\Theta : \mathbf{AlgInv}(K) \to \mathbf{Br}_h(K)$.

Proposition 2.27. *Let L/K be any field extension. Then the extension of scalars from K to L induces canonical symmetric monoidal functors that fit in a commutative square:*

$$\begin{array}{ccc} \mathbf{AlgInv}(K) & \xrightarrow{\Theta} & \mathbf{Br}_h(K) \\ \downarrow & & \downarrow \\ \mathbf{AlgInv}(L) & \xrightarrow{\Theta} & \mathbf{Br}_h(L). \end{array}$$

Proof. Everything is essentially clear, since tensor products are compatible with scalar extension. ☐

Proposition 2.28. *Let L/K be any field extension. Consider the following diagram (which does not commute):*

$$\begin{array}{ccc} \mathbf{Br}_h(K) & \longrightarrow & \mathbf{Br}_h(L) \\ & \searrow^{F} & \downarrow F \\ & & F(K)-\mathbf{ComAlg}_\Gamma \end{array}$$

where F is one of $\widetilde{SW}$, $\widetilde{GW}$ or $\widetilde{W}$, and $F(K)-\mathbf{ComAlg}_\Gamma$ is the category of Γ-graded commutative $F(K)$-(semi)-algebras.

Then this diagram commutes up to a natural transformation, which is given for any algebra with involution (A,σ) over K by a Γ-graded (semi-)ring morphism $F(A,\sigma) \to F(A_L,\sigma_L)$ induced by the extensions of scalars from K to L.

Proof. If $F = \widetilde{SW}$, the operations are preserved since direct sums and tensor products are compatible with scalar extension, and

$$\begin{array}{ccc} (A,\sigma)^{\otimes d} & \longrightarrow & (A_L,\sigma_L)^{\otimes d} \\ {\scriptstyle\varphi^{(d)}_{(A,\sigma)}}\downarrow & & \downarrow{\scriptstyle\varphi^{(d)}_{(A_L,\sigma_L)}} \\ (A,\sigma)^{\otimes r} & \longrightarrow & (A_L,\sigma_L)^{\otimes r}. \end{array}$$

commutes, seeing that the scalar extension of the bilinear space $(|A|_\sigma, T_\sigma)$ is $(|A_L|_{\sigma_L}, T_{\sigma_L})$.

By universal property of Grothedieck rings, this extends to a ring morphism for $F = \widetilde{GW}$, and also for $F = \widetilde{W}$ since scalar extension preserves hyperbolic modules.

The fact that this defines a natural transformation from the bottom to the top branch of the diagram follows from proposition 2.27, which shows that functoriality with respect to $\mathbf{Br}_h(K)$ is preserved by scalar extension. $\square$

If L/K is a finite field extension and $s : L \to K$ is any non-zero K-linear map, then it is well-known (see [9, 2.5.6]), that one can define a group morphism

$$\begin{array}{rccc} s_* : & GW(L) & \longrightarrow & GW(K) \\ & (V, b) & \longmapsto & (V, s \circ b) \end{array}$$

which induces a group morphism

$$s_* : W(L) \longrightarrow W(K),$$

and that these are respectively a $GW(K)$-module morphism and a $W(K)$-module morphism.

We wish to extend such a result to mixed Witt rings. Note that s extends naturally to a K-linear map $s : A_L \to A$ by $\mathrm{Id}_A \otimes s$, and this defines an *involution trace* in the sense of [5, 4.3]. This shows (see also [4, I.7.2,I.7.3.2]) that we can define Γ-graded group morphisms s_* that fit in a commutative diagram

$$\begin{array}{ccc} \widetilde{GW}(A_L, \sigma_L) & \xrightarrow{s_*} & \widetilde{GW}(A, \sigma) \\ \downarrow & & \downarrow \\ \widetilde{W}(A_L, \sigma_L) & \xrightarrow{s_*} & \widetilde{W}(A, \sigma). \end{array}$$

Proposition 2.29. *Let L/K be a finite field extension, and let $s : L \to K$ be a non-zero K-linear map. Then the scalar extension map $\rho : \widetilde{GW}(A, \sigma) \to \widetilde{GW}(A_L, \sigma_L)$ and the trace map $s_* : \widetilde{GW}(A_L, \sigma_L) \to \widetilde{GW}(A, \sigma)$ satisfy a reciprocity formula*

$$s_*(\rho(x) \cdot y) = x \cdot s_*(y) \tag{12}$$

for any $x \in \widetilde{GW}(A, \sigma)$ and $y \in \widetilde{GW}(A_L, \sigma_L)$.

In particular, if we see $\widetilde{GW}(A_L, \sigma_L)$ as a $\widetilde{GW}(A, \sigma)$-module through ρ, then s_* is a $\widetilde{GW}(A, \sigma)$-module morphism. Likewise, the map $s_* : \widetilde{W}(A_L, \sigma_L) \to \widetilde{W}(A, \sigma)$ is a $\widetilde{W}(A, \sigma)$-module morphism.

Proof. First note that the case of $\widetilde{W}(A, \sigma)$ follows easily from the case of $\widetilde{GW}(A, \sigma)$, using the fact that s_* preserves hyperbolic modules. Moreover, the statement about s_* being a module morphism is by definition equivalent to the reciprocity formula (12), and we only need to prove this formula for homogeneous elements in $\widetilde{SW}(A, \sigma)$ and $\widetilde{SW}(A, \sigma)$, so for $x \in SW^{\varepsilon_1}((A, \sigma)^{\otimes i_1})$ and $y \in SW^{\varepsilon_2}((A_L, \sigma_L)^{\otimes i_2})$ with $i_1, i_2 \in \{0, 1\}$.

In order to keep track of the tensor powers, we write $s_i : A_L^{\otimes i} \to A^{\otimes i}$ for the linear map induced by s, for any $i \in \mathbb{N}$ (in particular, $s_0 = s$). Then to conclude we need to show that the following outer diagram commutes:

$$
\begin{array}{ccc}
 & SW^{\varepsilon_1}((A,\sigma)^{\otimes i_1}) \otimes SW^{\varepsilon_2}((A_L,\sigma_L)^{\otimes i_2}) & \\
 \swarrow^{\rho\otimes 1} & & \searrow^{1\otimes (s_{i_2})_*} \\
SW^{\varepsilon_1}((A_L,\sigma_L)^{\otimes i_1}) \otimes SW^{\varepsilon_2}((A_L,\sigma_L)^{\otimes i_2}) & & SW^{\varepsilon_1}((A,\sigma)^{\otimes i_1}) \otimes SW^{\varepsilon_2}((A,\sigma)^{\otimes i_2}) \\
\downarrow & & \downarrow \\
SW^{\varepsilon_1\varepsilon_2}((A,\sigma)_L^{\otimes i_1+i_2}) & \xrightarrow{(s_{i_1+i_2})_*} & SW^{\varepsilon_1\varepsilon_2}((A,\sigma)^{\otimes i_1+i_2}) \\
\downarrow{\scriptstyle (\varphi^{(i_1+i_2)}_{(A_L,\sigma_L)})_*} & & \downarrow{\scriptstyle (\varphi^{(i_1+i_2)}_{(A,\sigma)})_*} \\
GW^{\varepsilon_1\varepsilon_2}((A,\sigma)_L^{\otimes i}) & \xrightarrow{(s_i)_*} & GW^{\varepsilon_1\varepsilon_2}((A,\sigma)^{\otimes i}).
\end{array}
$$

For the top pentagon, we can imitate the proof of the classical case given in [9, 2.5.6]. Indeed, take (V_1,h_1) a ε_1-hermitian module over $(A,\sigma)^{\otimes i_1}$, and (V_2,h_2) a ε_2-hermitian module over $(A,\sigma)_L^{\otimes i_2}$. Then the natural map

$$
\begin{array}{rcl}
(V_1 \otimes_K L) \otimes_L V_2 & \longrightarrow & V_1 \otimes_K V_2 \\
(x\otimes\lambda)\otimes y & \longmapsto & x \otimes (\lambda y)
\end{array}
$$

is an module isomorphism over

$$(A^{\otimes i_1} \otimes_K L) \otimes_L A_L^{\otimes i_2} \simeq A^{\otimes i_1} \otimes_K A_L^{\otimes i_2},$$

and we see that it is an isometry between $(\mathrm{Id}_A \otimes \mathrm{Id}_L \otimes s_{i_2}) \circ ((h_1\otimes 1)\otimes h_2)$ and $h_1 \otimes (s_{i_2} \circ h_2)$ under this identification.

The bottom square is tautological if i_1 or i_2 is zero, so we assume $i_1 = i_2 = 1$ and thus $i = 0$. We also set $\varepsilon = \varepsilon_1\varepsilon_2$. Let (V,h) be a ε-hermitian module over $(A_L,\sigma_L)^{\otimes 2}$. Then we have to find an isometry between the K-bilinear spaces (U,β) and (W,β') where $U = V\otimes_{A\otimes_K A} |A|_\sigma$ and $W = V \otimes_{A_L\otimes_L A_L} |A_L|_{\sigma_L}$, with

$$\beta(u\otimes a, v\otimes b) = \mathrm{Trd}_A\left(\sigma(a)(s_2(h(u,v))\cdot b)\right)$$

and

$$\beta'(u\otimes a\otimes\lambda, v\otimes b\otimes\mu) = s\left(\mathrm{Trd}_{A_L}((\sigma(a)\otimes\lambda)(h(u,v)\cdot(b\otimes\mu)))\right).$$

Now if we consider the following K-linear map

$$
\begin{array}{rrcl}
\Phi: & V\otimes_{A\otimes_K A} |A|_\sigma & \longrightarrow & V\otimes_{A_L\otimes_L A_L} |A_L|_{\sigma_L} \\
 & v\otimes a & \longmapsto & v\otimes(a\otimes 1),
\end{array}
$$

then

$$\beta'(\Phi(u\otimes a),\Phi(v\otimes b)) = s\left(\mathrm{Trd}_{A_L}((\sigma(a)\otimes 1)(h(u,v)\cdot(b\otimes 1)))\right).$$

We may then conclude that Φ is an isometry since it is easy to see that for any $x,y\in A$ and any $z\in A_L^{\otimes 2}$:

$$\mathrm{Trd}_A(x(s_2(z)\cdot y)) = s(\mathrm{Trd}_{A_L}(x_L(z\cdot y_L))).$$

Indeed, it is straightforward for $z = u\otimes v\otimes\lambda$ with $u,v\in A$ and $\lambda\in L$, so it follows for general z. $\square$

Remark 2.30. In particular, the image of the trace map s_* is an ideal in $\widetilde{GW}(A, \sigma)$ (resp. $\widetilde{W}(A, \sigma)$), which as in the classical case we call the *trace ideal* relative to L/K, and it can be shown that it does not depend on the choice of s.

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