Mixed Witt rings of algebras with involution of the first kind

Nicolas Garrel

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Introduction

In the 30s, Ernst Witt ([23]) started the algebraic study of quadratic forms over arbitrary fields (we will always assume that fields have characteristic not 2), as opposed to the previous more arithmetic-focused treatments. The key ingredient of his theory is to not only study individual quadratic forms, but consider them as a whole, and give the set of quadratic forms over some fixed field K (up to so-called *Witt equivalence)* a commutative ring structure $W(K)$, using direct sums and tensor products of quadratic spaces. The algebraic properties of this ring reflect many interesting properties of the underlying field and its quadratic forms; as an example that will be relevant to this article, the minimal prime ideals of $W(K)$ correspond to orderings of the field, the quotients giving the various signature maps. Depending on the situation, it may be more useful to consider the Grothendieck-Witt ring $GW(K)$, of which $W(K)$ is a quotient, which is arguably more fundamental (since it reflects isometries of quadratic forms and not only Witt equivalence), and has the additional structure of a λ-ring ([16, 9.10]), given naturally by the exterior powers of quadratic spaces.

These constructions can be somewhat extended to the framework of ε-hermitian forms over rings with involution (see section 1.1 or the reference [10]). If (A,σ) is a ring with involution (we always assume that 2 in invertible) We define the semi-group $SW^\varepsilon(A,\sigma)$ to be the set of isometry classes of regular ε-hermitian modules over (A,σ), with addition given by orthogonal direct sum. Its Grothendieck group is the Grothendieck-Witt group $GW^\varepsilon(A,\sigma)$ of (A,σ), and the quotient by the subgroup of hyperbolic spaces is the Witt group $W^\varepsilon(A,\sigma)$. We retrieve the Witt group of a field K by considering the ring with involution (K,Id) (and taking $\varepsilon = 1$).

One key complication that appears when A is not commutative is that there is no longer a good notion of tensor product of (hermitian) modules over (A,σ), since the tensor product of two modules over A is a module over $A\otimes_K A$, so we are left with only (Grothendieck-)Witt *groups* instead of *rings*. In general there is no obvious remedy to this, but the aim of this article is to study a special situation in which a work-around can be found. Specifically, suppose A is a central simple algebra over the field K, and σ is an involution of the first kind (so $\sigma_{|K} = \mathrm{Id}_K$). Then there is a hermitian Morita equivalence between $(A\otimes_K A, \sigma\otimes\sigma)$ and (K,Id), implying that there is a correspondance between hermitian modules over $(A\otimes_K A,\sigma\otimes\sigma)$ and quadratic modules over K, which is to say there is an isomorphism

$$GW(A \otimes_K A, \sigma \otimes \sigma) \xrightarrow{\sim} GW(K).$$

Thus the natural map

$$GW^{\varepsilon}(A, \sigma) \times GW^{\varepsilon}(A, \sigma) \to GW(A \otimes_K A, \sigma \otimes \sigma)$$

given by the tensor product over K actually defines a map

$$GW^{\varepsilon}(A, \sigma) \times GW^{\varepsilon}(A, \sigma) \to GW(K).$$

This allows us to construct an actual ($\mathbb{Z}/2\mathbb{Z}$-graded) ring

$$GW(K) \oplus GW^{\varepsilon}(A, \sigma).$$

If one wants to work with involutions of both types (orthogonal and symplectic) at once, it is much more convenient to consider all at once hermitian *and* anti-hermitian forms over (A, σ), and hence rather construct what we call the mixed Grothendieck-Witt ring of (A, σ) (definition 2.16):

$$\widetilde{GW}(A, \sigma) = GW(K) \oplus GW^{-}(K) \oplus GW(A, \sigma) \oplus GW^{-}(A, \sigma),$$

as well as its quotient the mixed Witt ring (definition 2.21):

$$\widetilde{W}(A, \sigma) = W(K) \oplus W(A, \sigma) \oplus W^{-}(A, \sigma),$$

which are commutative rings, naturally graded over the Klein group.

Overview

Section 1 is dedicated to hermitian Morita theory, as exposed for instance in [10]. Our point of view is that the theory is most conveniently expressed in the context of a certain monoidal category $\mathbf{Br}_h(K)$ (inspired by works such as [5] in the non-hermitian case) which we call the hermitian Brauer 2-group (see proposition 1.7). After a brief review of the basic theory of algebras with involution and hermitian modules, we define this category, and check that it is a coherent 2-group (see definition 1.12 or [4]). We then introduce the notion of strongly 2-torsion coherent 2-group, and show that $\mathbf{Br}_h(K)$ satisfies this definition (theorem 1.15), using in a crucial way the so-called involution trace form and the Goldman element (which we study in section 1.2).

In section 2, we define the mixed (Grothendieck-)Witt ring of an algebra with involution, and study some of its basic properties. We find it more convenient to first work with a semi-group $\widetilde{SW}(A, \sigma)$. Using the properties of $\mathbf{Br}_h(K)$, we show that it is a commutative semi-ring (proposition 2.7), and that it is functorial in (A, σ) with respect to the category $\mathbf{Br}_h(K)$, which expresses compatibility with Morita equivalence (proposition 2.12). We deduce the same results for $\widetilde{GW}(A, \sigma)$ (theorem 2.18) and $\widetilde{W}(A, \sigma)$ (theorem 2.22), which constitute the main results of the article. In addition, we study the effect of scalar extension (section 2.5), the case where $(A, \sigma) = (K, \mathrm{Id})$ (section 2.6), and products of diagonal forms (proposition 2.28).

Section 3 is dedicated to some computations in $\widetilde{W}(A, \sigma)$ when (A, σ) is a crossed product with involution (in a similar but slightly more general setting than [9], but less general than for instance [19]). We give a brief review of Galois algebras (section 3.1) and crossed products (section 3.2), before we describe a simple parametrization of crossed products with involution (proposition 3.11). We use this to compute some products in $\widetilde{W}(A,\sigma)$ (corollary 3.14), and in particular any product when A is a quaternion algebra (proposition 3.22).

In section 4 we study exterior powers of hermitian modules, which is not an obvious notion since A is not commutative. Adapting ideas found in [11] (following Tamagawa), we define the alterning powers $\mathrm{Alt}^d(V)$ of a module V (definition 4.1), which inherit a hermitian form $\mathrm{Alt}^d(h)$ from (V,h) (definition 4.9). Using a certain Morita equivalence, we then define a new hermitian module $(\Lambda^d(V),\lambda^d(h))$, and the central result of the section is that this defines a pre-λ-ring structure on $\widetilde{GW}(A,\sigma)$, functorial in (A,σ) (theorem 4.30). In section 4.5 we suggest a somewhat different approach of the classical notion of determinant of an involution (definition 4.36, to be compared with [11, 7.2]), and show a natural duality result (corollary 4.45). We also discuss some open questions, in particular whether $\widetilde{GW}(A,\sigma)$ is a λ-ring.

Section 5 is a brief overview of the basic properties of the fundamental filtration of the mixed Witt ring, and its associated graded ring, which we call the mixed cohomology ring by analogy with Milnor's conjecture. The ulterior goal is to construct cohomological invariants of algebras with involution in a natural way.

Finally, in section 6 we use the ring structure on $\widetilde{W}(A,\sigma)$ to give a more natural treatment to the theory of signatures of involutions and hermitian forms (for involutions of the first kind), as developped in [2] and [3] (following older ideas in the literature). In particular we give a full description of $\mathrm{Spec}(\widetilde{W}(A,\sigma))$ (proposition 6.4), very similar to the classical case of $W(K)$. Above each ordering of K there are two distinct signature maps on $\widetilde{W}(A,\sigma)$, and we interpret the results in [3] as a quest to find a reasonable choice of signature above each ordering, which is captured by a simple topological condition (theorem 6.32).

Preliminaries and conventions

We fix a base field K of characteristic not 2, and we identify symmetric bilinear forms and quadratic forms over K, through $b \mapsto q_b$ with $q_b(x) = b(x,x)$. Diagonal quadratic forms are denoted $\langle a_1,\dots,a_n\rangle$, with $a_i \in K^*$, and $\langle\langle a_1,\dots,a_n\rangle\rangle$ is the n-fold Pfister form $\langle 1,-a_1\rangle\cdots\langle 1,-a_n\rangle$.

All rings are associative and with unit, and ring morphisms preserve the units. Unless otherwise specified, modules are by default modules on the *right*, and are assumed to be faithful modules. Every algebra and every module have finite dimension over K. If A is a K-algebra, and V is a right A-module, then $\mathrm{End}_A(V)$ is a K-algebra acting on V on the left, with the tautological action of functions on V. On the other hand, if V is a A-module on the *left*, then we endow $\mathrm{End}_A(V)$ with the product *opposite* to usual function composition, so that $\mathrm{End}_A(V)$ acts on V on the *right*.

When we say that (A,σ) is an algebra with involution over K, we mean that A is a central simple algebra over K, and that σ is an involution of the first kind on A, so σ is an anti-automorphism of K-algebra of A, with $\sigma^2 = \mathrm{Id}_A$. In general, "involution" will be synonym with "involution of the first kind". We set $\mathrm{Sym}(A,\sigma)$ for the set of symmetric elements of A, which satisfy $\sigma(a) = a$, and $\mathrm{Skew}(A,\sigma)$ for the set of anti-symmetric elements, for which $\sigma(a) = -a$. The involution σ is orthogonal if $\dim_K(\mathrm{Sym}(A,\sigma)) = n(n+1)/2$, and it is symplectic if $\dim_K(\mathrm{Sym}(A,\sigma)) = n(n-1)/2$. In particular, (K, Id) is an algebra with orthogonal involution. A quaternion algebra admits a unique symplectic involution, called its canonical involution, and we denote it by γ.

If A is a central simple algebra over K, we write $\mathrm{Trd}_A : A \to K$ for the reduced trace of A, and $\mathrm{Nrd}_A : A \to K$ for its reduced norm. They can be defined as descents of the usual trace and determinant maps on endormorphism algebras of vector spaces.

If L/K is any field extension, and X is an object (algebra, module, involution, hermitian form, etc.) over K, then $X_L = X \otimes_K L$ is the corresponding object over L, obtained by base change.

A semi-group is a set endowed with an associative binary product (so the difference with a monoid is the existence of a unit). If Γ is a monoid, then a semi-group S is Γ-graded if it is equipped with a direct sum decomposition $S = \bigoplus_{g\in\Gamma} S_g$. The Grothencieck group $G(S)$ of S is the universal solution to the problem of finding a morphism $S \to G$ where G is a group. In the case where S is a commutative semi-group with cancellation (meaning that $x + y = x + z$ implies $y = z$), then the construction of $G(S)$ is exactly the same as that of $\mathbb{Z}$ from $\mathbb{N}$, and the structural morphism $S \to G(S)$ is injective. Precisely, elements of $G(S)$ are formal differences $x - y$ with $x, y \in S$, and $x - y = x' - y'$ in $G(M)$ iff $x + y' = x' + y$ in S. Since the functor $S \mapsto G(S)$ preserves direct sums, if S is Γ-graded then $G(S)$ inherits a natural Γ-grading.

What we call a semi-ring is a triple $(S, +, \cdot)$ such that $(S, +)$ is a commutative semi-group, $(S, \cdot)$ is a monoid, and we have the distributive law (often in the literature it is asked that $(S, +)$ is a monoid). We say that S is Γ-graded if $(S, +)$ has a Γ-grading such that $S_g S_h \subset S_{gh}$ for all $g, h \in \Gamma$. If S is a semi-ring then $G(S)$ is naturally a ring, and if S is Γ-graded then $G(S)$ is a Γ-graded ring.

If G is a group and R is a G-graded ring, then for any group H the group algebra $R[H]$ is naturally a $(G \times H)$-graded ring. The augmentation map $\varepsilon : R[H] \to R$ is a G-graded ring morphism.

If R is a commutative ring, $\mathrm{Spec}(R)$ is the associated affine scheme, $\mathrm{Spec}_0(R)$ is the generic fiber of $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{Z})$ (which is the space of prime ideals of residual characteristic 0), and $\mathrm{minSpec}(R)$ is the subspace of minimal prime ideals.

1 The hermitian Brauer 2-group

In this section we review hermitian Morita theory, as developped in [7] or [10], in the case of algebras with involution (for which we take [11] as a reference). We adopt a categorical point of view that allows the theory to be expressed in a very efficient way. The idea that (non-hermitian) Morita theory can be expressed as the definition of some 2-category of algebras and bimodules has been explored for instance in [5], but as far as we know this is the first time the hermitian analogue is written down explicitly, even though it is mostly a matter of reformulation. In fact, we define a certain category $\mathbf{Br}_h(K)$, which we call the *hermitian Brauer 2-group* of the field K, and much of the classical hermitian Morita theory can be expressed by saying that $\mathbf{Br}_h(K)$ is a "coherent 2-group". Furthermore, we introduce the notion of a "strongly 2-torsion" coherent 2-group, and show that $\mathbf{Br}_h(K)$ satisfies this condition, which will be crucial in the rest of the article, notably for proving the associativity of our various rings.

1.1 Hermitian modules and involutions

We start, for reader's convenience as well as for establishing notations, by reviewing basic facts about hermitian modules, all stated without proof (see [10] for a comprehensive reference over general rings with involution). Let A be a central simple algebra over K, and let V be a right A-module. Then $B = \mathrm{End}_A(V)$ is a central simple algebra over K that is Brauer-equivalent to A (this may be taken as a definition of the Brauer-equivalence), and there is a canonical identification $A \simeq \mathrm{End}_B(V)$ where V is naturally viewed as a left B-module. Such a B-A-bimodule will be called a *Morita* bimodule. In this situation, $\mathrm{Hom}_A(V, A)$ and $\mathrm{Hom}_B(V, B)$ are both naturally A-B-bimodule, and are actually canonically isomorphic: $f \in \mathrm{Hom}_B(V, B)$ corresponds to $f' \in \mathrm{Hom}_A(V, A)$ such that for any $x, y \in V$:

$$f(x)y = xf'(y).$$

We will identify those two bimodules, and use the common notation $V^\vee$; then $V^\vee$ is a Morita bimodule. There are also natural identifications $A \simeq V^\vee \otimes_B V$ and $B \simeq V \otimes_A V^\vee$. As usual, there is a canonical bidual isomorphism $V \simeq V^{\vee\vee}$, which is a bimodule isomorphism. The *reduced dimension* of V is $\mathrm{rdim}_A(V) = \deg(B)$.

Suppose A and B are endowed with respective involutions σ and τ. Then we can define a A-B-bimodule $\overline{V}$ by $axb = \tau(b)x\sigma(a)$ for all $a \in A$, $b \in B$ and $x \in V$. It is obviously a Morita bimodule whenever V is, and clearly $\overline{\overline{V}} = V$. Thus we have two methods for swapping the algebras of a Morita bimodule, and it turns out that they commute, so we may define the *adjoint* bidomule of V as $V^* = \overline{V}^\vee = \overline{V^\vee}$, which is a Morita B-A-bimodule. It is easily seen that $V \mapsto V^*$ defines a contravariant endofunctor on Morita B-A-bimodules, such that there is a natural isomorphism

$$\begin{array}{ccc} V & \xrightarrow{\sim} & V^{**} \\ x & \longmapsto & \sigma \circ ev_x \end{array}$$

where ev_x is the evaluation of linear forms at x.

Note that if we do not specify τ, we can still define $\overline{V}$ as a left A-module, and A^* as a right A-module. Then $\mathrm{End}_A(V^*) \simeq B^{op}$, so we need the choice of an involution τ to turn V^* into a B-A-bimodule. Given the data of (A, σ) and V, we define a *sequilinear form* h on V over (A, σ) to be a biadditive map $h : V \times V \to A$ such that for all $x, y \in V$ and all $a, b \in A$:

$$h(xa, yb) = \sigma(a)h(x, y)b.$$

This is equivalent to defining a right A-module morphism $\hat{h} : V \to V^*$, where h and $\hat{h}$ are related by $h(x, y) = \hat{h}(x)(y)$. We say that h is *regular* if $\hat{h}$ is a module isomorphism. Applying the "adjoint" functor to $\hat{h}$, we get a natural map $V^{**} \to V^*$, and under the identification $V \simeq V^{**}$ we get $\hat{h}' : V \to V^*$. We say that h is ε-hermitian for some $\varepsilon \in K^*$ when $\hat{h} = \varepsilon\hat{h}'$. This is equivalent to the formula

$$h(y, x) = \varepsilon\sigma(h(x, y))$$

for all $x, y \in V$. It is easily seen that $\varepsilon^2 = 1$, so $\varepsilon = \pm 1$. We say that h is hermitian if $\varepsilon = 1$, and anti-hermitian if $\varepsilon = -1$ (if $(A, \sigma) = (K, \mathrm{Id})$, we are defining symmetric and anti-symmetric bilinear forms), and we call ε the *sign* of h (sometimes denoted ε_h).

Now assume that B also has an involution τ, so V^* is a B-A-bimodule. Then we say that (V, h) is a ε-hermitian Morita bimodule between (B, τ) and (A, σ) if h is ε-hermitian and $\hat{h}$ is actually an isomorphism of *bi*modules. The additional condition on τ amounts to $\tau = \sigma_h$ where σ_h is the so-called *adjoint involution* on B defined by

$$h(bx, y) = h(x, \sigma_h(b)y)$$

for all $b \in B$, $x, y \in V$. So in the same way that a right A-module defines a Morita bimodule for a unique B, a regular ε-hermitian right module over (A, σ) defines a ε-hermitian Morita bimodule for a unique (B, τ). Recall that an involution can have two types: orthogonal or symplectic. Then σ and τ have the same type if $\varepsilon = 1$, and opposite type if $\varepsilon = -1$. We define the *type* of h (or (V, h)) to be the type of τ. We often identify the type of h or τ with an element of ± 1, with $+1$ corresponding to the orthogonal type, and -1 to the symplectic type. Thus to h are associated its sign and its type in $\{\pm 1\}$, and they coincide iff σ is orthogonal.

When (V, h) is a Morita ε-hermitian bimodule between (B, τ) and (A, σ), then there is a natural ε-hermitian form $\overline{h} : \overline{V} \times \overline{V} \to B$ over (B, τ) such that $\hat{\overline{h}} : \overline{V} \to \overline{V}^*$ is $\overline{\hat{h}}$ (using $\overline{V}^* = \overline{V^*}$). Then $\overline{h}$ is characterized by

$$\overline{h}(x, y)z = xh(y, z).$$

Note that every construction and every result stated above is compatible with a change of base field.

1.2 The Goldman element

A very crucial feature of central simple algebras that will prove extremely useful for us is the existence of the so-called *Goldman element*, and we take the time here to collect some of its properties than are relevant in the article. We base our account on [11, 3.A,10.A].

It is a defining property of Azumaya algebras that the so-called "sandwich action"

$$\begin{array}{rcl} A \otimes_K A^{op} & \longrightarrow & \mathrm{End}_K(A) \\ a \otimes b & \longmapsto & (x \mapsto axy) \end{array} \qquad (1)$$

is a K-algebra isomorphism. In particular, there is an element in $g_A \in A \otimes_K A$, the Goldman element of A, that corresponds to the reduced trace of A, seen as a linear map $A \to K \subset A$. If A is split, so $A = \mathrm{End}_K(V)$ for some K-vector space V, then $g_A \in \mathrm{End}_K(V \otimes_K V)$ is the exchange map $(x \otimes y \mapsto y \otimes x)$ ([11, 3.6]).

More generally, for any $d \in \mathbb{N}$, there is a unique group morphism $\mathfrak{S}_d \to (A^{\otimes d})^*$ ([11, 10.1]), which we will denote as $\pi \mapsto g_A(\pi)$, such that

$$g_A((i\; i+1)) = 1 \otimes \cdots \otimes 1 \otimes g_A \otimes 1 \otimes \cdots \otimes 1.$$

For $d = 0, 1$ this is the trivial morphism, and for $d = 2$ the morphism property just means that $g_A^2 = 1$.

Lemma 1.1. *Let A and B be central simple algebras over K, and let V be a Morita B-A-module. Then for any $v_1, \dots, v_d \in V$ and any $\pi \in \mathfrak{S}_d$:*

$$g_B(\pi) \cdot (v_1 \otimes \cdots \otimes v_d) = (v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(d)}) \cdot g_A(\pi).$$

Remark 1.2. In particular, when $d = 2$ and $A = B = V$, we get that for any $x, y \in A$ we have $(x \otimes y)g_A = g_A(y \otimes x)$, which is shown in [11, 3.6].

Moreover, if $A = K$, so $B = \mathrm{End}_K(V)$ is split, this means that $g_B(\pi)$ is the K-linear map $(v_1 \otimes \cdots \otimes v_d \mapsto v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(d)})$, which is shown in [11, 10.1].

Proof. By construction of $g_A(\pi)$ and $g_B(\pi)$, we can reduce to the case where $d = 2$ and π is the transposition, and extending the scalars if necessary we may assume that A and B are split. In this case we have $A \simeq \mathrm{End}_K(U)$, $B \simeq \mathrm{End}_K(W)$, and $V \simeq \mathrm{Hom}_K(U, W)$ with obvious actions, and for any $f_1, f_2 \in V$ and $u_1, u_2 \in U$:

$$\begin{aligned}(g_B \cdot f_1 \otimes f_2)(u_1 \otimes u_2) &= g_B(f_1(u_1) \otimes f_2(u_2)) \\ &= f_2(u_2) \otimes f_1(u_1) \\ &= (f_2 \otimes f_1)(g_A(u_1 \otimes u_2)) \\ &= (f_2 \otimes f_1 \cdot g_A)(u_1 \otimes u_2)\end{aligned}$$

so indeed $g_B \cdot f_1 \otimes f_2 = f_2 \otimes f_1 \cdot g_A$. □

Now suppose A is endowed with an involution σ. Then we may twist the above sandwich action (1) to

$$\begin{array}{rcl} A \otimes_K A & \longrightarrow & \mathrm{End}_K(A) \\ a \otimes b & \longmapsto & (x \mapsto ax\sigma(y)). \end{array} \tag{2}$$

We will call this the "twisted sandwich action", and unless otherwise specified this is always the action we have in mind when we consider A as a left $(A \otimes_K A)$-module (of course it depends on the choice of an involution σ). The following lemma is inspired by [11, exercise I.12].

Lemma 1.3. *The Goldman element of A is symmetric for the involution $\sigma \otimes \sigma$ on $A \otimes_K A$, for any choice of involution σ of the first kind on A.*

Furthermore, under the twisted sandwich action defined by σ, we find for any $x \in A$, with $\varepsilon = \pm 1$ the type of σ:

$$\begin{aligned} g_A \cdot x &= \varepsilon\sigma(x), \\ (\mathrm{Id} \otimes \sigma)(g_A) \cdot x &= \mathrm{Trd}_A(x), \\ (\sigma \otimes \mathrm{Id})(g_A) \cdot x &= \mathrm{Trd}_A(x), \\ (\sigma \otimes \sigma)(g_A) \cdot x &= \varepsilon\sigma(x). \end{aligned}$$

Proof. For the first statement see [11, 10.19]. As for the equalities, the second one is a reformulation of the definition of g_A, and the third and fourth follow from the two first using that $(\sigma \otimes \sigma)(g_A) = g_A$. It just remains to prove the first one.

We may reduce to the case where A is split, so $A = \mathrm{End}_K(V)$ for some K-vector space V, and $\sigma = \sigma_b$ where b is a ε-symmetric bilinear form. We then have the canonical isomorphism $A \simeq V \otimes V$, such that $(v \otimes w)(v' \otimes w') = b(w, v')v \otimes w'$ and $\sigma(v \otimes w) = \varepsilon w \otimes v$. Let (e_i) be a basis of V, and (e_i^*) its dual basis for b (if σ is orthogonal, we may take for (e_i) an orthogonal basis, in which case $e_i = e_i^*$). Then $g = \sum_{i,j} e_i \otimes e_j^* \otimes e_j \otimes e_i^*$ (see the proof of [11, 3.6]), so

$$\begin{aligned} g_A \cdot (v \otimes w) &= \sum_{i,j} (e_i \otimes e_j^*) \cdot (v \otimes w) \cdot \varepsilon(e_i^* \otimes e_j) \\ &= \varepsilon \sum_{i,j} b(e_j^*, v) b(w, e_i^*) e_j \otimes e_j \\ &= \left(\sum_i b(e_i^*, w) e_i \right) \otimes \left(\sum_j b(e_j^*, v) e_j \right) \\ &= w \otimes v. \end{aligned}$$

□

1.3 Definition of $\mathbf{Br}_h(K)$

We now come to the definition of the category $\mathbf{Br}_h(K)$. The objects are the algebras with involution (A, σ) over K. A morphism from (B, τ) to (A, σ) is an isometry class of Morita ε-hermitian bimodule. We will usually identify a module and its isometry class when no confusion is possible. To properly define a category structure, we have to specify how to compose morphisms, and look for identity morphisms.

Definition 1.4. *Let (A, σ) be an algebra with involution over K. If $a \in A^*$ is a ε-symmetric invertible element, meaning that $\sigma(a) = \varepsilon a$, then we define a ε-hermitian form $\langle a \rangle_\sigma$ on A, seen as a right A-module:*

$$\begin{array}{ccc} A \times A & \longrightarrow & A \\ (x, y) & \longmapsto & \sigma(x) a y \end{array}.$$

This definition in particular makes sense for $a \in K^*$ and $\varepsilon = 1$.

We will write $\langle a_1, \dots, a_n \rangle_\sigma$ for an orthogonal sum $\langle a_1 \rangle_\sigma \perp \cdots \perp \langle a_n \rangle_\sigma$ (where all a_i are ε-symmetric), and call such a form diagonal.

Remark 1.5. If A is a division algebra, then any ε-hermitian form is diagonal in this sense, but this is not the case in general.

Proposition-definition 1.6. *Let (A, σ), (B, τ) and (C, θ) be algebras with involution over K. Let (U, h) be a ε-hermitian Morita bimodule between (C, θ) and (B, τ), and let (V, h') be a ε'-hermitian Morita bimodule between (B, τ) and (A, σ).*

Then $(V, h') \circ (U, h) = (V \circ U, h' \circ h)$ is a $\varepsilon\varepsilon'$-hermitian Morita bimodule between (C, θ) and (A, σ), where

$$V \circ U = U \otimes_B V$$

and

$$(h' \circ h)(u \otimes v, u' \otimes v') = h'(v, h(u, u')v'). \tag{3}$$

Moreover, the isometry class of $(V \circ U, h' \circ h)$ only depends on the isometry classes of (U, h) and (V, h').

Proof. This is a special case of [10, I.8.1]. □

We can now state:

Proposition 1.7. *With the composition of morphisms being given by definition 1.6,* $\mathbf{Br}_h(K)$ *is a groupoid. The identity of* (A,σ) *is* $(A,\langle 1\rangle_\sigma)$*, where* A *is a* A*-*A*-bimodule in the tautological way, and the inverse of a morphism* (V,h) *is* $(\overline{V},\varepsilon_h\overline{h})$ *(where* h *is* ε_h*-hermitian).*

Proof. The associativity of the composition is proved in [10, lemma I.8.1.1]. The statement on identities is immediate given the definition of the composition, and the statement about inverses is proved in [10, prop I.9.3.4]. □

Remark 1.8. The automorphisms in $\mathbf{Br}_h(K)$ of an object (A,σ) are exactly the diagonal forms $\langle a\rangle_\sigma$ for $a\in K^*$.

Remark 1.9. If f is a morphism in $\mathbf{Br}_h(K)$, then we have defined in section 1.1 both its *sign* and its *type*. For composable morphisms, the sign of $f\circ g$ is the product of the signs of f and g, while the type of $f\circ g$ is simply the type of g.

Remark 1.10. There are several ways to extend this definition. First, it would be natural to define a (weak) *2-category* instead of a category, where the morphisms would be actual bimodules (instead of isometry classes), and 2-morphisms would be isometries. Then $\mathbf{Br}_h(K)$ is the 1-truncature of this 2-category, which would logically be called the hermitian Brauer 3-group.

We may also allow the base field K to carry itself an involution ι, and consider algebras with involution that restrict to ι on K. This would allow a treatment of unitary involutions when ι is non-trivial, and the present construction could correspond to $\iota=\mathrm{Id}_K$. The analogue of mixed Witt rings for unitary involutions is more involved, and will be the subject of a future article.

We could extend the construction to more general base rings, instead of just fields, and even to base schemes or potentially to locally ringed topos, in the vein of [6].

Finally, we may consider more general algebras than Azumaya algebras, which would give a larger (2-)category, and then retrieve $\mathbf{Br}_h(K)$ as the subcategory of "weakly invertible" objects.

All these directions can be combined for maximal generality, but are not useful for the purpose of this article: we only need to consider isometry classes for our constructions, and we require invertible objects and involutions of the first kind. The extension to more general bases should be the subject of a future article, but the case of base fields is already interesting.

Remark 1.11. It is known that a central simple algebra has an involution of the first kind iff its Brauer class is in the 2-torsion subgroup $\mathrm{Br}(K)[2]$. Furthermore, if (A,σ) and (B,τ) are such that A and B are Brauer-equivalent, so that there is a Morita B-A-bimodule V, then there is a ε-hermitian form h on V (unique up to similitude) such that $\sigma_h=\tau$. In other words, there is a morphism between two objects (B,τ) and (A,σ) in $\mathbf{Br}_h(K)$ iff A and B are Brauer-equivalent. In particular, the set of isomorphism classes in $\mathbf{Br}_h(K)$ is canonically identified with $\mathrm{Br}(K)[2]$.

1.4 The 2-group structure

The category $\mathbf{Br}_h(K)$ inherits a monoidal structure from the usual tensor product of algebras and modules. This structure has remarkable features of invertibility and symmetry, which we encapsulate in theorem 1.15. The relevant framework is that of *2-groups*. Like many higher categorical matters, this notion is subject to many slightly different definitions, depending on the authors; we base our account on [4].

Recall that a weakly monoidal category is a category C endowed with a functor $\otimes : C \times C \to C$ and a unit object 1, and isomorphisms

$$\begin{aligned} \alpha_{x,y,z} &: (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z) \\ l_x &: 1 \otimes x \xrightarrow{\sim} x \\ r_x &: x \otimes 1 \xrightarrow{\sim} x \end{aligned}$$

verifying coherence conditions, most notably the MacLane pentagon (see [15, VII.1]). The category is symmetric if moreover one has isomorphisms

$$s_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x$$

satisfying some other coherence conditions, most importantly that $s_{y,x} \circ s_{x,y} = \mathrm{Id}_{x\otimes y}$ (see [15, VII.7]).

It is a basic fact that if (V, h) and (V', h') are respectively a ε-hermitian module over (A, σ) and a ε'-hermitian module over (A', σ'), then $(V \otimes_K V', h\otimes h')$ is a $\varepsilon\varepsilon'$-hermitian module over $(A \otimes_K A', \sigma \otimes \sigma')$, with $\mathrm{End}_{A\otimes_K A'}(V \otimes_K V') \simeq \mathrm{End}_A(V) \otimes_K \mathrm{End}_{A'}(V')$, and $\sigma_{h\otimes h'} = \sigma_h \otimes \sigma_{h'}$. Combined with elementary facts about tensor products of algebras and modules, this easily establishes that $\mathbf{Br}_h(K)$ is a symmetric weakly monoidal category, with (K, Id) as a unit.

To account for (weak) invertibility of objects, we use the following definition:

Definition 1.12. *A a coherent 2-group is a weakly monoidal category $(C, \otimes, 1)$ where every morphism is invertible, and every object $x \in C$ is equipped with an adjoint equivalence $(x, \overline{x}, i_x, e_x)$, meaning that we are given an object $\overline{x}$ (the weak inverse of x) and isomorphisms $i_x : 1 \to x \otimes \overline{x}$ and $e_x : \overline{x} \otimes x \to 1$ such that the following diagram commutes:*

$$\begin{array}{ccccc} 1 \otimes x & \xrightarrow{i_x\otimes 1} & (x \otimes \overline{x}) \otimes x & \xrightarrow{\alpha_{x,\overline{x},x}} & x \otimes (\overline{x} \otimes x) \\ \downarrow{\scriptstyle l_x} & & & & \downarrow{\scriptstyle 1\otimes e_x} \\ x & & \xrightarrow{\quad r_x^{-1} \quad} & & x \otimes 1. \end{array}$$

To simplify notations, we will assume that C is strictly monoidal (which is a benign assumption thanks to the coherence theorems, see [15, VII.2]). Then the diagram simply states that the two obvious morphisms $x \otimes \overline{x} \otimes x \to x$ obtained by collapsing either the left or the right side of the product actually give the same map.

In [4], it is shown that $x \mapsto \overline{x}$ can be made into a covariant functor, but we will be interested in a case where this is trivially true. In fact, since two objects of $\mathbf{Br}_h(K)$ are isomorphic iff the underlying algebras are Brauer-equivalent (see remark 1.11), the weak inverse of (A, σ) must be some (B, τ) such that $A \otimes_K B$ is Brauer equivalent to K, meaning that $[A] + [B] = 0$ in $\mathrm{Br}(K)$. But since

$[A]$ has order at most 2, this means that A and B must be Brauer-equivalent. The most obvious choice is then $(B,\tau) = (A,\sigma)$, and we will show that indeed this works. By analogy with the case of ordinary groups, where the property that $x^{-1} = x$ correspond to being a 2-torsion group, we make the following definition:

Definition 1.13. *A coherent 2-group is said to be 2-torsion if any object x is its own weak inverse, so $\overline{x} = x$.*

If C is a 2-torsion coherent 2-group, then for any object x there are two isomorphisms $x \otimes x \to 1$ given by the structure, namely e_x and i_x^{-1}. This leads to the following definition:

Definition 1.14. *A coherent 2-group is said to be strongly 2-torsion if for any object x, we have $\overline{x} = x$ and $i_x = e_x^{-1}$.*

Following the definition, if $(C,\otimes,1)$ is a monoidal category (which as before we treat as if it were strictly monoidal), it is a strongly 2-torsion coherent 2-group if it is a groupoid and for every objet x we give an isomorphism $e_x : x \otimes x \to 1$ such that the two natural isomorphisms $x \otimes x \otimes x \to x$, obtained by cancelling either the first two or the last two terms, are the same.

We may then state:

Theorem 1.15. *The category $\mathbf{Br}_h(K)$ is a strongly 2-torsion coherent 2-group, with the counit*

$$(A \otimes_K A, \sigma \otimes \sigma) \to (K, \mathrm{Id})$$

of the adjunction associated to (A,σ) given by the hermitian bimodule (A, T_σ), where the left action of $A \otimes_K A$ on A is the "twisted sandwich action" (see (2)) and T_σ is the so-called involution trace form:

$$T_\sigma(x,y) = \mathrm{Trd}_A(\sigma(x)y). \tag{4}$$

Proof. We first have to check that (A, T_σ) indeed defines a morphism from $(A \otimes_K A, \sigma \otimes \sigma)$ to (K, Id) in $\mathbf{Br}_h(K)$; this means that $\sigma \otimes \sigma$ is the adjoint involution of T_σ, which is shown in [11, prop 11.1]. Then we have to check that the two morphisms $(A^{\otimes 3}, \sigma^{\otimes 3}) \to (A,\sigma)$ obtained by collapsing either the first two or last two factors are actually equal, or otherwise formulated, that the following diagram commutes :

$$\begin{array}{ccc} (A \otimes_K A \otimes_K A, \sigma \otimes \sigma \otimes \sigma) & \longrightarrow & (K \otimes A, \mathrm{Id} \otimes \sigma) \\ \downarrow & & \downarrow \\ (A \otimes K, \sigma \otimes \mathrm{Id}) & \longrightarrow & (A,\sigma) \end{array}$$

which amounts to giving an isometry between the hermitian bimodules $(A \otimes_K A, h_1)$ and $(A \otimes_K A, h_2)$ where the left action of $A \otimes_K A \otimes_K A$ and the right action of A are in the first case

$$(a \otimes b \otimes c) \cdot_1 (x \otimes y) \cdot_1 d = (ax\sigma(b)) \otimes (cyd)$$

and in the second case

$$(a \otimes b \otimes c) \cdot_2 (x \otimes y) \cdot_2 d = (axd) \otimes (by\sigma(c))$$

and the hermitian forms are respectively

$$h_1(x\otimes y, x'\otimes y') = \operatorname{Trd}_A(\sigma(x)x')\sigma(y)y'$$

and

$$h_2(x\otimes y, x'\otimes y') = \operatorname{Trd}_A(\sigma(y)y')\sigma(x)x'.$$

Let $g\in A\otimes_K A$ be the Goldman element; if $g=\sum_i a_i\otimes b_i$, we define our isometry from h_1 to h_2 by

$$\begin{array}{rccc} \varphi: & A\otimes_K A & \longrightarrow & A\otimes_K A \\ & x\otimes y & \longmapsto & \sum_i (xa_iy)\otimes\sigma(b_i). \end{array}$$

Note that $1\otimes 1$ is sent to $g'=(\operatorname{Id}\otimes\sigma)(g)$. Since the bimodule $A\otimes_K A$ is generated by the element $1\otimes 1$ for both actions defined above, this is enough to characterize φ, but we need to check that g' is an admissible element for the image of $1\otimes 1$, which is equivalent to checking that we indeed defined a bimodule morphism. We have

$$\begin{aligned} \varphi\left((a\otimes b\otimes c)\cdot_1 (x\otimes y)\cdot_1 d\right) &= \varphi\left((ax\sigma(b))\otimes(cyd)\right) \\ &= \sum_i (ax\sigma(b)a_icyd)\otimes\sigma(b_i) \end{aligned}$$

and

$$\begin{aligned} (a\otimes b\otimes c)\cdot_2 \varphi(x\otimes y)\cdot_2 d &= \sum_i (a\otimes b\otimes c)\cdot_2 (xa_iy)\otimes\sigma(b_i)\cdot_2 d \\ &= \sum_i (axa_iyd)\otimes(b\sigma(b_i)\sigma(c)). \end{aligned}$$

We are thus led to show that

$$\sum_i \sigma(b)a_ic\otimes\sigma(b_i) = \sum_i a_i\otimes b\sigma(b_i)\sigma(c),$$

which amounts to

$$(\operatorname{Id}\otimes\sigma)((\sigma(b)\otimes 1)g(c\otimes 1)) = (\operatorname{Id}\otimes\sigma)((1\otimes c)g(1\otimes\sigma(b))).$$

Now this is a consequence of lemma 1.1. Hence φ indeed defines a bimodule morphism, and since g' is invertible it is an isomorphism.

We still have to verify that φ is an isometry from h_1 to h_2. We have

$$\begin{aligned} h_2(\varphi(x\otimes y),\varphi(x'\otimes y')) &= \sum_{i,j} h_2((xa_iy)\otimes\sigma(b_i),(x'a_jy')\otimes\sigma(b_j)) \\ &= \sum_{i,j} \operatorname{Trd}_A(b_i\sigma(b_j))\sigma(y)\sigma(a_i)\sigma(x)x'a_jy' \\ &= \sum_{i,j,k} \sigma(y)\sigma(a_i)\sigma(x)x'a_jy'a_kb_i\sigma(b_j)b_k \\ &= \varepsilon\sum_{i,k} \sigma(y)\sigma(a_i)\sigma(x)x'\sigma(b_i)\sigma(a_k)\sigma(y')b_k \\ &= \varepsilon\sigma(y)\operatorname{Trd}_A(\sigma(x)x')\varepsilon y' \\ &= h_1(x\otimes y, x'\otimes y') \end{aligned}$$

using the relations in lemma 1.3. □

1.5 Associativity in a 2-torsion coherent 2-group

We show that the sort of associativity we get in a strongly 2-torsion coherent 2-group when considering $x \otimes x \otimes x \to x$ can be generalized to an arbitrary number of terms. Let P be the free unital (non-associative) magma on a single object a. There is a unique magma morphism $P \to \mathbb{N}$ sending a to 1, and the fiber above some $d \in \mathbb{N}$ is denoted $P(d)$. An element of $P(d)$ is called a *bracketing* on d objects, and can be seen as a choice of order in which to multiply d elements using a non-necessarily associative binary product. For instance, $P(d)$ has only one element for $d = 0, 1, 2$, while $P(3)$ has 2 elements, namely $a(aa)$ and $(aa)a$. For any $d \geqslant 2$ and any $B \in P(d)$, there are unique $d_i \in \mathbb{N}^*$ and $B_i \in P(d_i)$ (for $i = 1, 2$), such that $d_1 + d_2 = d$ and $B = B_1 B_2$.

For any object x in a 2-torsion coherent 2-group C, any $d \in \mathbb{N}$ and any $B \in P(d)$, we define an isomorphism

$$\varphi_x^B : x^{\otimes d} \longrightarrow \begin{cases} x & \text{if } d \text{ is odd} \\ 1 & \text{if } d \text{ is even} \end{cases} \tag{5}$$

inductively on d: if $d = 0, 1$ then φ_x^B is the appropriate identity morphism for the only $B \in P(d)$. If $B \in P(d)$ for $d \geqslant 2$, then we write $B = B_1 B_2$. If d_1 or d_2 is even, then we set

$$\varphi_x^B = \varphi_x^{B_1} \otimes \varphi_x^{B_2}. \tag{6}$$

If d_1 and d_2 are odd, we set

$$\varphi_x^B = e_x \circ (\varphi_x^{B_1} \otimes \varphi_x^{B_2}). \tag{7}$$

In particular, $\varphi_x^B = e_x$ when $B \in P(2)$.

Then we can rephrase the definition of strong 2-torsion as the fact that for any object x, φ_x^B is independent of the choice of B for the two possible $B \in P(3)$. In general, we get:

Proposition 1.16. *Let C be a strongly 2-torsion coherent 2-group. Then for any $d \in \mathbb{N}$, the isomorphism φ_x^B is independent of the choice of $B \in P(d)$.*

Proof. For any $d \in \mathbb{N}$, we write $L_d \in P(d)$ for the left-associated bracketing on d elements, corresponding to $(\cdots((aa)a)\cdots)a)$. It may be defined inductively by $L_{d+1} = L_d \cdot L_1$ (obviously L_0 is the empty bracketing, and $L_1 = a$). We write $\varphi_x^{(d)} = \varphi_x^{L_d}$.

We prove by induction on d that $\varphi_x^B = \varphi_x^{(d)}$, similarly to how the generalized associativity law is proved in a semi-group. For $d \leqslant 2$ there is only one bracketing, so the statement is obvious. Now fix some $d \geqslant 3$ and assume the statement is true for any $r < d$. Let B be any bracketing on d objects, and write $B = B_1 B_2$, with $B_i \in P(d_i)$, $d_i \in \mathbb{N}^*$. Then by induction hypothesis we have $\varphi_x^{B_i} = \varphi_x^{(d_i)}$.

Suppose first that d_1 is even. Then $\varphi_x^B = \varphi_x^{(d_1)} \otimes \varphi_x^{(d_2)}$. If d_2 is odd, we get

$$\begin{aligned}
\varphi_x^{(d_1)} \otimes \varphi_x^{(d_2)} &= \varphi_x^{(d_1)} \otimes \varphi_x^{(d_2-1)} \otimes \varphi_x^{(1)} \\
&= \varphi_x^{L_{d_1} \cdot L_{d_2-1}} \otimes \varphi_x^{(1)} \\
&= \varphi_x^{(d-1)} \otimes \varphi_x^{(1)} \\
&= \varphi_x^{(d)}
\end{aligned}$$

using the induction hypothesis for $d-1$, so we are done. If on the other hand d_2 is even:

$$\begin{aligned}\varphi_x^{(d_1)} \otimes \varphi_x^{(d_2)} &= \varphi_x^{(d_1)} \otimes e_x \circ (\varphi_x^{(d_2-1)} \otimes \varphi_x^{(1)}) \\ &= e_x \circ (\varphi_x^{(d_1)} \otimes \varphi_x^{(d_2-1)} \otimes \varphi_x^{(1)}) \\ &= e_x \circ (\varphi_x^{L_{d_1} \cdot L_{d_2-1}} \otimes \varphi_x^{(1)}) \\ &= e_x \circ (\varphi_x^{(d-1)} \otimes \varphi_x^{(1)}) \\ &= \varphi_x^{(d)}.\end{aligned}$$

Now suppose d_1 is odd. If d_2 is odd, then

$$\begin{aligned}\varphi_x^B &= e_x \circ (\varphi_x^{(d_1)} \otimes \varphi_x^{(d_2)}) \\ &= e_x \circ (\varphi_x^{(d_1)} \otimes \varphi_x^{(d_2-1)} \otimes \varphi_x^{(1)}) \\ &= e_x \circ (\varphi_x^{L_{d_1} \cdot L_{d_2-1}} \otimes \varphi_x^{(1)}) \\ &= e_x \circ (\varphi_x^{(d-1)} \otimes \varphi_x^{(1)}) \\ &= \varphi_x^{(d)}.\end{aligned}$$

Finally, if d_2 is even:

$$\begin{aligned}\varphi_x^B &= \varphi_x^{(d_1)} \otimes \varphi_x^{(d_2)} \\ &= \varphi_x^{(d_1)} \otimes e_x \circ (\varphi_x^{(d_2-1)} \otimes \varphi_x^{(1)}) \\ &= e_x \circ (\varphi_x^{(d_1)} \otimes \varphi_x^{(d_2-1)}) \otimes \varphi_x^{(1)} \\ &= \varphi_x^{L_{d_1} \cdot L_{d_2-1}} \otimes \varphi_x^{(1)} \\ &= \varphi_x^{(d-1)} \otimes \varphi_x^{(1)} \\ &= \varphi_x^{(d)}\end{aligned}$$

where we finally use that $1 \otimes e_x = e_x \otimes 1$, which is the characterization of strong 2-torsion. $\square$

Given this result, we can make the following definition:

Definition 1.17. *In a strongly 2-torsion coherent 2-group, we write $\varphi_x^{(d)}$ for φ_x^B where B is any bracketing on d objects.*

1.6 Additional properties

We now show a few properties of $\mathbf{Br}_h(K)$ that will be important for the construction of the mixed Witt ring. First note that there is an additional structure on $\mathbf{Br}_h(K)$ that does not fit in the definition of a 2-group: for any two morphisms f and f' in $\mathbf{Br}_h(K)$ having the same target (A,σ) and having the same sign (so they both correspond to ε-hermitian modules for the same ε), then there is a natural notion of sum $f \oplus f$, corresponding to the orthogonal direct sum of ε-hermitian modules.

Lemma 1.18. *Let f, f' be two morphisms in $\mathbf{Br}_h(K)$ having the same target (B,τ) and of the same sign, and let $g : (B,\tau) \to (A,\sigma)$. Then*

$$(f \oplus f') \circ g = (f \circ g) \oplus (f' \circ g).$$

Proof. This is immediate since the composition of morphisms in $\mathbf{Br}_h(K)$ is defined using the tensor product, which is distributive over direct sums. □

There is also a symmetry property that will later imply the commutativity of our rings.

Proposition 1.19. *If A be a central simple algebra over K, then for any right A-modules V_1 and V_2, there is a canonical $(A \otimes_K A)$-module isomorphism*

$$\begin{array}{rccc} \Sigma : & V_1 \otimes_K V_2 & \xrightarrow{\sim} & V_2 \otimes_K V_1 \\ & v_1 \otimes v_2 & \longmapsto & (v_2 \otimes v_1) g_A \end{array}$$

such that for any involution σ on A, and any ε_i-hermitian form h_i on V_i (with respect to σ), Σ is an isometry of $\varepsilon_1\varepsilon_2$-hermitian forms over $(A \otimes_K A, \sigma \otimes \sigma)$ from $h_1 \otimes h_2$ to $h_2 \otimes h_1$.

Proof. The fact that Σ is a module morphism follows from lemma 1.1 since $g_A(x \otimes y) = (y \otimes x)g_A$. Now to show that it defines an isometry:

$$\begin{aligned} (h_2 \otimes h_1)(\Sigma(v_1 \otimes v_2), \Sigma(v_1' \otimes v_2')) &= (h_2 \otimes h_1)((v_2 \otimes v_1)g_A, (v_2' \otimes v_1')g_A) \\ &= g_A h_2(v_2, v_2') \otimes h_1(v_1 \otimes v_1')g_A \\ &= h_1(v_1, v_1') \otimes h_2(v_2, v_2') \end{aligned}$$

using that g_A is symmetric for $\sigma \otimes \sigma$ (see lemma 1.3) and lemma 1.1. □

Finally, there is an obvious compatibility with base change:

Proposition 1.20. *Let L/K be any field extension. Then the association $(A, \sigma) \mapsto (A_L, \sigma_L)$ and $(V, h) \mapsto (V_L, h_L)$ defines a monoidal functor $\mathbf{Br}_h(K) \to \mathbf{Br}_h(L)$, which sends $\varphi^{(d)}_{(A,\sigma)}$ to $\varphi^{(d)}_{(A_L,\sigma_L)}$.*

Proof. Algebras with involution and ε-hermitian modules are compatible with base change, as is the composition in $\mathbf{Br}_h(K)$ since it is defined with a tensor product, and likewise for the monoidal structure. Note that the involution trace form is also preserved by base change, which implies the statement about $\varphi^{(d)}_{(A,\sigma)}$. □

1.7 A zero object

Up until now, we have excluded zero modules from our exposition. Indeed, they lack the good invertibility properties we have discussed so far. For instance, take A a central simple algebra over K, and consider the zero module $V = \{0\}$. Then $B = \mathrm{End}_A(V)$ is the zero ring, so obviously do not get $A \simeq \mathrm{End}_B(V)$.

On the other hand, it will be convenient at times to be able to consider zero modules. Notably, we will construct some kind of exterior power operation on modules, and like in the case of vector spaces these powers vanish above the module's dimension. For this reason, we define a slightly bigger category $\mathbf{Br}_h(K)'$, which is obtained from $\mathbf{Br}_h(K)$ by formally adding a zero object, and a zero morphism between any two objects (the composition of a zero morphism with any morphism being the appropriate zero morphism). The zero object can be interpreted as the zero ring with its trivial involution, and the zero morphisms are zero bimodules with the trivial hermitian form (which can be given either sign). Obviously these are not Morita bimodules, and $\mathbf{Br}_h(K)'$ is not a groupoid.

We will explicitly state in the remaining of the article if we allow the zero ring and zero morphisms in a statement.

2 The mixed Witt ring of an algebra with involution

We now want to use the hermitian Morita theory developped in the first section to define a product such that $\widetilde{GW}(A,\sigma)$ and $\widetilde{W}(A,\sigma)$ are commutative graded rings, as described in the introduction.

2.1 The mixed Witt semi-ring

If one wants to define the usual Witt and Grothendieck-Witt rings of a field K, it makes sense to start by defining some semi-ring $SW(K)$ (which we may call the Witt semi-ring of K), saying that $SW(K)$ is the set of isometry classes of symmetric (non-degenerated) bilinear forms over K, endowed with orthogonal direct sums, and tensor products. Then the Grothendieck-Witt ring $GW(K)$ is the Grothendieck ring of $SW(K)$, and $W(K)$ is the quotient of $GW(K)$ by the ideal of hyperbolic forms.

We can also define $SW^-(K)$ as the additive semi-group of isometry classes of anti-symmetric bilinear forms over K, with Grothendieck group $GW^-(K)$. Then $SW^\pm(K) = SW(K) \oplus SW^-(K)$ is a semi-ring, with Grothendieck ring $GW^\pm(K) = GW(K) \oplus GW^-(K)$. This ring scarcely appears in the litterature since $GW^-(K)$ is very uninteresting over fields of characteristic not 2, but we will need it for our construction. The corresponding quotient by hyperbolic forms is still $W(K)$ (we may write that $W^-(K) = 0$). Obviously $SW^\pm(K)$ is graded over $\mathbb{Z}/2\mathbb{Z}$, and the grading is inherited by $GW^\pm(K)$.

We want to adapt these constructions to the context of algebras with involutions.

Definition 2.1. *Let (A,σ) be an algebra with involution over K. Then we define for $\varepsilon = \pm 1$ the commutative additive semi-group $SW^\varepsilon(A,\sigma)$ to be the set of isometry classes of regular ε-hermitian modules over (A,σ), with the orthogonal direct sum. We often write $SW(A,\sigma)$ for $SW^+(A,\sigma)$. We also write $SW_\varepsilon(A,\sigma)$ for the isometry classes of hermitian modules of type ε (so $SW^\varepsilon(A,\sigma) = SW_\varepsilon(A,\sigma)$ iff σ is orthogonal).*

We then set

$$\begin{aligned} SW^\pm(A,\sigma) &= SW(A,\sigma) \oplus SW^-(A,\sigma) = SW_+(A,\sigma) \oplus SW_-(A,\sigma) \\ \widetilde{SW}(A,\sigma) &= SW^\pm(K) \oplus SW^\pm(A,\sigma). \end{aligned}$$

Remark 2.2. The set $SW(A,\sigma) \cup SW^-(A,\sigma)$ is the set of morphisms to (A,σ) in $\mathbf{Br}_h(K)$ (so in other words it is the set of objects of the slice category $\mathbf{Br}_h(K)_{/(A,\sigma)}$). There is a natural way to define the sum of elements of $SW(A,\sigma)$ and $SW^-(A,\sigma)$, namely the orthogonal sum as sesquilinear spaces, but we do not want to use this operation since the resulting sum is neither hermitian nor anti-hermitian, and thus does not have an adjoint involution (or to put it differently it is not a morphism in $\mathbf{Br}_h(K)$).

Remark 2.3. We can also define $SW^\varepsilon(A,\sigma)' = SW^\varepsilon(A,\sigma)\cup\{0\}$, the additive monoid obtained by formally adding a neutral element. Then $SW(A,\sigma)'\cup SW^-(A,\sigma)'$ is the set of morphisms to (A,σ) in the larger category $\mathbf{Br}_h(K)'$. Since want to avoid zero modules as much as possible we usually use $SW^\varepsilon(A,\sigma)$, which is just a semi-group instead of a monoid.

We already have a structure of semi-ring on $SW^\pm(K)$, and the usual tensor product induces an obvious structure of $SW^\pm(K)$-module on $\widetilde{SW}(A,\sigma)$. To make $\widetilde{SW}(A,\sigma)$ a semi-ring, it is then enough to define the product of an element of $SW^\varepsilon(A,\sigma)$ and an element of $SW^{\varepsilon'}(A,\sigma)$.

Definition 2.4. *Let (A,σ) be a central simple algebra with involution of the first kind over K, and for $i=1,2$ let $(V_i,h_i)\in SW^{\varepsilon_i}(A,\sigma)$. Then we define*

$$(V_1,h_1)\cdot(V_2,h_2) = (V_1\cdot V_2, h_1\cdot h_2)\in SW^{\varepsilon_1\varepsilon_2}(K)$$

by the following composition in $\mathbf{Br}_h(K)$:

$$(V_1\cdot V_2, h_1\cdot h_2) = (A,T_\sigma)\circ(V_1\otimes_K V_2, h_1\otimes h_2).$$

Remark 2.5. The K-vector space $V_1\cdot V_2$ is independent of h_1 and h_2, but it does depend on σ. Precisely, unwrapping the definition 1.6, we have

$$V_1\cdot V_2 = (V_1\otimes_K V_2)\otimes_{A\otimes_K A} A \tag{8}$$

where the left action of $A\otimes_K A$ on A is given by the twisted sandwich action through σ (see 2).

Remark 2.6. Still unwrapping 1.6, and using formula (3) for the composition of hermitian forms, and (4) for the definition of T_σ, we find the explicit formula

$$(h_1\cdot h_2)(u_1\otimes u_2\otimes a, v_1\otimes v_2\otimes b) = \mathrm{Trd}_A\left(\sigma(a)h_1(u_1,v_1)b\sigma(h_2(u_2,v_2))\right). \tag{9}$$

Now we have defined a product $x\cdot y$ for any x and y that are elements of one of the four components of $\widetilde{SW}(A,\sigma)$, whch gives by obvious extension by bilinearity a binary product on $\widetilde{SW}(A,\sigma)$.

Proposition 2.7. *Equipped with the product defined above, $\widetilde{SW}(A,\sigma)$ is a commutative semi-ring.*

Proof. For commutativity, clearly it is enough to show $x\cdot y = y\cdot x$ when x and y are in one of the four components of $\widetilde{SW}(A,\sigma)$. If x or y is in $SW^\pm(K)$ then this is clear by the basic properties of the tensor product (or if we want to give a more fancy answer, it follows from the fact that (K,Id) is the unit element in the monoidal category $\mathbf{Br}_h(K)$). When $x\in SW^{\varepsilon_1}(A,\sigma)$ and $y\in SW^{\varepsilon_2}(A,\sigma)$, it is a direct consequence of proposition 1.19.

For distributivity, once again it is clearly enough to consider to show that $x(y+z) = xy+xz$ when x, y and z are in components of $\widetilde{SW}(A,\sigma)$ (with y and z in the same component). Again, the result is obvious if one of those components is $SW(K)$ or $SW^-(K)$ by the usual properties of tensor products, and in the remaining cases it is an immediate corollary of lemma 1.18.

It remains to show associativity. Here also we are readily reduced to showing $x_1(x_2x_3) = (x_1x_2)x_3$ where $x_i\in SW^{\varepsilon_i}(A,\sigma)$. Now this follows from the fact that $\mathbf{Br}_h(K)$ is strongly 2-torsion (theorem 1.15). Indeed, $(x_1 \otimes x_2) \otimes x_3 \simeq x_1 \otimes (x_2 \otimes x_3)$ as $\varepsilon_1\varepsilon_2\varepsilon_3$-hermitian modules over $(A^{\otimes 3}, \sigma^{\otimes 3})$, and $x_1(x_2x_3)$ and $(x_1x_2)x_3$ are obtained by composing this morphism in $\mathbf{Br}_h(K)$ with either $\varphi^B_{(A,\sigma)}$ or $\varphi^{B'}_{(A,\sigma)}$ where B and B' are the two bracketings on 3 objects; but as we observed just before the proof of proposition 1.16, the characterization of strong 2-torsion is precisely that $\varphi^B_{(A,\sigma)} = \varphi^{B'}_{(A,\sigma)}$, so indeed the product is associative.

□

2.2 Functoriality and grading

Given any morphism $f : (B,\tau) \to (A,\sigma)$ is $\mathbf{Br}_h(K)$, there is a natural induced map $f_* : \widetilde{SW}(B,\tau) \to \widetilde{SW}(A,\sigma)$, which is the identity on the $SW^{\pm}(K)$ component, and which is the composition with f on the components $SW(B,\tau)$ and $SW^-(B,\tau)$, recalling that their elements are by definition morphisms in $\mathbf{Br}_h(K)$ with destination (B,τ).

Example 2.8. We can describe the action of f_* on elementary diagonal forms, that is forms of the type $\langle a\rangle_\tau$ for $a \in B^*$ ε-symmetric. Suppose f represents the module (V,h) over (A,σ). Then $f_*(\langle a\rangle_\tau)$ has underlying space $B \otimes_B V$, which is canonically V, and the hermitian form is

$$\begin{array}{rcl} V \times V & \longrightarrow & A \\ (x,y) & \longmapsto & h(x,ay). \end{array}$$

Remark 2.9. In particular, $f_*(\langle 1\rangle_\tau) = f$. Thus the semi-rings $\widetilde{SW}(A,\sigma)$ all have a distinguished element $\langle 1\rangle_\sigma \in SW(A,\sigma)$, and any element of any $SW^\varepsilon(A,\sigma)$ can be put in correspondence with some $\langle 1\rangle_\tau$ for some (B,τ) by an appropriate f_* (and actually f is given by the element itself).

Example 2.10. Now suppose $f = \langle a\rangle_\sigma$ is itself diagonal. Then it is a morphism $(A,\sigma_a) \to (A,\sigma)$ in $\mathbf{Br}_h(K)$, where $\sigma_a(x) = a^{-1}\sigma(x)a$. If (V,h) is a ε-hermitian module over (A,σ_a), then $f_*(V,h)$ has underlying module $V \otimes_A A \simeq V$, and the form is

$$\begin{array}{rcl} V \times V & \longrightarrow & A \\ (x,y) & \longmapsto & h(xa,y). \end{array}$$

There is an obvious $\mathbb{Z}/2\mathbb{Z}$-grading on the semi-ring $\widetilde{SW}(A,\sigma)$, where the *even* component is $SW(K)^{\pm}$, and the *odd* component is $SW^{\pm}(A,\sigma)$. Now there are two natural ways to extend this to a Γ-grading where $\Gamma = (\mathbb{Z}/2\mathbb{Z})^2$ is the Klein group. On $SW^{\pm}(K)$, there is not really a choice since $SW(K)$ must be the neutral component. But on $SW^{\pm}(A,\sigma)$, we can attribute the grading according to the *sign* or to the *type*.

Notice that according to remark 1.9, if the sign of $f : (B,\tau) \to (A,\sigma)$ is ε, then $f_*(SW^{\varepsilon'}(B,\tau)) = SW^{\varepsilon\varepsilon'}(A,\sigma)$, while on the other hand $f_*(SW_{\varepsilon'}(B,\tau)) = SW_{\varepsilon'}(A,\sigma)$. So if we want f_* to preserve the Γ-gradings, we have to choose the type grading (the sign grading is only preserved by hermitian morphisms, and not anti-hermitian ones). The component corresponding to $(1,0) \in \Gamma$ will then be the orthogonal component (namely $SW_+(A,\sigma)$), and the component corresponding to $(1,1) \in \Gamma$ will be the symplectic one (namely $SW_-(A,\sigma)$).

So our choice of grading is such that f_* is a morphism of Γ-graded semi-groups, and we want to show that it is actually a morphism of semi-rings.

Before that, we show an intermediary result which is stronger than we need for now but will be useful later on. Recall from 1.17 the definition of $\varphi^{(d)}_{(A,\sigma)}$.

Proposition 2.11. *Let $f : (B,\tau) \longrightarrow (A,\sigma)$ be a morphism in $\mathbf{Br}_h(K)$. For any $d \in \mathbb{N}$ we have a commutative square in $\mathbf{Br}_h(K)$:*

$$\begin{array}{ccc} (B^{\otimes d}, \tau^{\otimes d}) & \xrightarrow{f^{\otimes d}} & (A^{\otimes d}, \sigma^{\otimes d}) \\ \varphi^{(d)}_{(B,\tau)} \downarrow & & \downarrow \varphi^{(d)}_{(A,\sigma)} \\ (B,\tau) & \xrightarrow{\quad f \quad} & (A,\sigma) \end{array}$$

if d is odd, and

$$\begin{array}{ccc} (B^{\otimes d}, \tau^{\otimes d}) & \xrightarrow{f^{\otimes d}} & (A^{\otimes d}, \sigma^{\otimes d}) \\ \varphi^{(d)}_{(B,\tau)} \downarrow & & \downarrow \varphi^{(d)}_{(A,\sigma)} \\ (K,\mathrm{Id}) & \xrightarrow{\quad \mathrm{Id} \quad} & (K,\mathrm{Id}) \end{array}$$

if d is even.

Proof. If $d = 0$, all four arrows are the identity, and if $d = 1$ the horizontal arrows are the same, while the vertical arrows are the identity. From the construction of $\varphi^{(d)}_{(A,\sigma)}$, it is easy to see that the case where $d \geqslant 2$ reduces by induction to $d = 2$, meaning that we have to construct an isometry between the modules corresponding to $\varphi^{(2)}_{(A,\sigma)} \circ f^{\otimes 2}$ and $\varphi^{(2)}_{(A,\sigma)}$.

If f corresponds to the ε-hermitian module (V,h), then we define the following $(B \otimes_K B)$-K-bimodule morphism:

$$\begin{array}{rrcl} \psi : & (V \otimes_K V) \otimes_{A \otimes_K A} A & \longrightarrow & B \\ & (v \otimes w) \otimes a & \longmapsto & \varphi_h(va \otimes w). \end{array}$$

where $\varphi_h : V \otimes_K V \to B$ corresponds to the canonical identification $V \otimes_A V^* \to B$ (see section 1.1 or [11, 5.1]), and is given, identifying $B = \mathrm{End}_A(V)$, by

$$\varphi_h(v \otimes w)(x) = vh(w,x).$$

Then ψ is well-defined since for $x, y \in A$:

$$\begin{aligned} \psi((v \otimes w) \otimes (xa\sigma(y)) &= \varphi_h(vxa\sigma(y) \otimes w) \\ &= \varphi_h(vxa \otimes wy) \\ &= \psi((vx \otimes wy) \otimes a), \end{aligned}$$

and it is a bimodule morphism since for $x, y \in B$:

$$\begin{aligned} \psi((xv \otimes yw) \otimes a) &= \varphi_h(xva \otimes yw) \\ &= x\varphi_h(va \otimes v)\tau(y) \end{aligned}$$

because the canonical identification $V \otimes_A V^* \simeq B$ is a B-B-bimodule isomorphism.

To show that ψ is an isometry, we must establish equality between on the one hand

$$\begin{aligned} &\mathrm{Trd}_B\left(\tau\left(\psi((v \otimes w) \otimes a)\right) \cdot \psi(((v' \otimes w') \otimes b))\right) \\ &= \mathrm{Trd}_B\left(\tau(\varphi_h(va \otimes w)) \cdot \varphi_h(v'b \otimes w')\right) \end{aligned}$$

and on the other hand

$$\begin{aligned}&\operatorname{Trd}_A\left(\sigma(a)(h\otimes h)(v\otimes w, v'\otimes w')\cdot b\right)\\ &=\varepsilon\operatorname{Trd}_A(\sigma(a)h(v,v')bh(w',w)).\end{aligned}$$

Now applying successively the formulas in theorem [11, 5.1], we get:

$$\begin{aligned}&\operatorname{Trd}_B\left(\tau(\varphi_h(va\otimes w))\cdot\varphi_h(v'b\otimes w')\right)\\ &=\varepsilon\operatorname{Trd}_B\left(\varphi_h(w\otimes va)\cdot\varphi_h(v'b\otimes w')\right)\\ &=\varepsilon\operatorname{Trd}_B\left(\varphi_h(wh(va,v'b)\otimes w')\right)\\ &=\varepsilon\operatorname{Trd}_A\left(h(w',wh(va,v'b))\right)\\ &=\varepsilon\operatorname{Trd}_A\left(h(w',w)\sigma(a)h(v,v')b\right).\end{aligned}$$

□

We can finally state:

Proposition 2.12. *The association $(A,\sigma)\mapsto\widetilde{SW}(A,\sigma)$ and $f\mapsto f_*$ defines a functor*

$$\widetilde{SW}:\mathbf{Br}_h(K)\longrightarrow\mathbf{ComSemRing}_\Gamma$$

where $\mathbf{ComSemRing}_\Gamma$ is the category of commutative Γ-graded semi-rings.

Proof. Since f_* is defined by composition with f, it is clear that $\widetilde{SW}$ is at least a functor to the category of sets. The fact that f_* preserves the sum is a direct consequence of lemma 1.18, and the fact that it preserves the Γ-grading has been discussed above.

Thus it remains to show that f_* is compatible with the product. As always we can reduce to the case of homogeneous elements, and we have to prove that for $f:(B,\tau)\to(A,\sigma)$ having sign ε, the following diagram commutes:

$$\begin{array}{ccc} SW^{\varepsilon_1}(B_1,\tau_1)\times SW^{\varepsilon_2}(B_2,\tau_2) & \longrightarrow & SW^{\varepsilon_1'}(A_1,\sigma_1)\times SW^{\varepsilon_2'}(A_2,\sigma_2)\\ \downarrow & & \downarrow\\ SW^{\varepsilon_1\varepsilon_2}(B_1\otimes_K B_2,\tau_1\otimes\tau_2) & \longrightarrow & SW^{\varepsilon_1'\varepsilon_2'}(A_1\otimes_K A_2,\sigma_1\otimes\sigma_2)\\ \downarrow & & \downarrow\\ SW^{\varepsilon_1\varepsilon_2}(B_3,\tau_3) & \longrightarrow & SW^{\varepsilon_1'\varepsilon_2'}(A_3,\sigma_3).\end{array}$$

where for $i=1,2,3$ we have either $(B_i,\tau_i)=(B,\tau)$ and $(A_i,\sigma_i)=(A,\sigma)$, and in this case $\varepsilon_i'=\varepsilon\varepsilon_i$, or $(B_i,\tau_i)=(A_i,\sigma_i)=(K,\mathrm{Id})$, and in this case $\varepsilon_i'=\varepsilon_i$. The horizontal maps are the obvious ones depending on the case, induced by f.

Now the top square always commutes because of basic properties of the tensor product, and the bottom square is easily seen to commute if one of the B_i is K. So we need to show that the bottom square commutes in the following configuration:

$$\begin{array}{ccc} SW^{\varepsilon_1\varepsilon_2}(B\otimes_K B,\tau\otimes\tau) & \longrightarrow & SW^{\varepsilon_1\varepsilon_2}(K)\\ \downarrow & & \downarrow\\ SW^{\varepsilon_1\varepsilon_2}(A\otimes_K A,\sigma\otimes\sigma) & \longrightarrow & SW^{\varepsilon_1\varepsilon_2}(K)\end{array}$$

which is a direct consequence of proposition 2.11. □

Example 2.13. Since $\widetilde{SW}$ is a functor, any automorphism of (A,σ) in $\mathbf{Br}_h(K)$ induces an automorphism of $\widetilde{SW}(A,\sigma)$; we call such an automorphism *standard*. Recall from remark 1.8 that the automorphisms of (A,σ) are the $\langle\lambda\rangle_\sigma$ where $\lambda \in K^*$. Then according to example 2.10, the associated standard automorphism is the identity on $SW^{\pm}(K)$, and the multiplication by $\langle\lambda\rangle$ on $SW^{\pm}(A,\sigma)$. It is easy to check directly that this is indeed an automorphism.

Remark 2.14. Since $\mathbf{Br}_h(K)$ is a groupoid, this means that all induced maps f_* are isomorphisms. Now seeing that (A,σ) and (B,τ) are isomorphic in $\mathbf{Br}_h(K)$ iff A and B are Brauer-equivalent, we can conclude that $\widetilde{SW}(A,\sigma)$ only depends on the Brauer class of A, up to a graded semi-ring isomorphism. Thus we may if necessary reduce to the study of division algebras with involution: if D is the division algebra equivalent to A, and if θ is any involution on D, then $\widetilde{SW}(A,\sigma) \approx \widetilde{SW}(D,\theta)$; but note that the isomorphism is non-canonical. Precisely, it is determined up to a standard automorphism of $\widetilde{SW}(D,\theta)$.

Remark 2.15. This functorial behaviour, which is very useful, is the main justification for the definition of a four-component mixed Witt semi-ring. It might have been more intuitive to define either $\widetilde{SW}^{\varepsilon}(A,\sigma) = SW(K) \oplus SW^{\varepsilon}(A,\sigma)$ or $\widetilde{SW}_{\varepsilon}(A,\sigma) = SW(K) \oplus SW_{\varepsilon}(A,\sigma)$. But in the first case, we only get functoriality with respect to hermitian morphisms, so in particular the ring depends not only on the Brauer class of A but also on the type of σ. In the second case, we do get a full functoriality, and the ring only depends on $[A]$, but on the other hand it prevents us from working with hermitian forms over any kind of involutions (we have to work with hermitian *or* anti-hermitian forms according to the type of the involution). This 4-component semi-ring is a good compromise which affords full freedom, despite being a little more cumbersome.

2.3 The mixed Grothendieck-Witt ring

Now that we have defined the mixed Witt semi-ring of an algebra with involution, we may use it to define the rings that are of more direct interest to us.

Note that Witt's cancellation theorem holds over algebras with involution, which means that $\widetilde{SW}(A,\sigma)$ is an additive semi-group with cancellation (see [10, 6.3.4] for the case of division algebras, which is enough by remark 2.14).

Definition 2.16. *Let (A,σ) be an algebra with involution over K. Then the mixed Grothendieck-Witt ring $\widetilde{GW}(A,\sigma)$ is the Grothendieck ring of $\widetilde{SW}(A,\sigma)$.*

We also write $GW^{\varepsilon}(A,\sigma)$ and $GW_{\varepsilon}(A,\sigma)$ for the Grothendieck groups of $SW^{\varepsilon}(A,\sigma)$ and $SW_{\varepsilon}(A,\sigma)$ respectively, so that

$$\begin{aligned}\widetilde{GW}(A,\sigma) &= GW(K) \oplus GW^-(K) \oplus GW(A,\sigma) \oplus GW^-(A,\sigma)\\ &= GW(K) \oplus GW^-(K) \oplus GW_+(A,\sigma) \oplus GW_-(A,\sigma).\end{aligned}$$

Remark 2.17. In [13], Lewis makes a very similar construction in the case where $A = Q$ is a quaternion algebra and $\sigma = \gamma$ is its canonical involution. His definition is essentially the same, except that he uses the norm form of Q instead of the involution trace form. Since the two forms are the same up to a factor $\langle 2\rangle$, this yields naturally isomorphic rings. However, the norm form is a special feature of quaternion algebras (in general for an algebra of degree n the reduced norm is a homogeneous polynomial function of degree n), so the construction does not generalize well. Furthermore, no proof of the associativity of the product is given in [13] (even though it is far from obvious).

Thus the mixed Grothendieck-Witt ring $\widetilde{GW}(A,\sigma)$ is a commutative Γ-graded ring. From the formal properties of Grothendieck rings and proposition 2.12, we easily deduce:

Theorem 2.18. *The functor $\widetilde{SW}$ extends naturally to a functor*

$$\widetilde{GW} : \mathbf{Br}_h(K) \longrightarrow \mathbf{ComRing}_\Gamma$$

where $\mathbf{ComRing}_\Gamma$ is the category of Γ-graded commutative rings.

As before, this implies in particular that up to (non-canonical) isomorphism, $\widetilde{GW}(A,\sigma)$ only depends on the Brauer class of A. Likewise, $\widetilde{GW}_\varepsilon(A,\sigma) = GW(K) \oplus GW_\varepsilon(A,\sigma)$ is a commutative $\mathbb{Z}/2\mathbb{Z}$-graded ring, that is functorial in (A,σ) (relative to the category $\mathbf{Br}_h(K)$), and thus only depends on $[A]$ up to isomorphism.

Remark 2.19. Given the pre-existing $GW^\pm(K)$-module structure on $\widetilde{GW}(A,\sigma)$, we could have defined its ring structure as the unique one such that $\widetilde{GW}$ is a functor from $\mathbf{Br}_h(K)$ to Γ-graded $GW^\pm(K)$-algebras with $\langle 1\rangle_\sigma^2 = T_\sigma$.

2.4 The mixed Witt ring

To define the mixed Witt ring, we first need to discuss hyperbolic forms. Since we assumed that $char(K) \neq 2$, this is the same as metabolic forms, so we may characterize a hyperbolic ε-hermitian module (V,h) by the existence of a submodule $U \subset V$ such that $U^\perp = U$ (which is called a *Lagrangian* of (V,h)). In particular, any element of $SW^-(K)$ is hyperbolic, as is well known. We call an element of $\widetilde{SW}(A,\sigma)$ hyperbolic if each of its homogeneous component is hyperbolic, and an element of $\widetilde{GW}(A,\sigma)$ is hyperbolic if it is the difference of two hyperbolic elements of $\widetilde{SW}(A,\sigma)$.

Proposition 2.20. *Let (A,σ) be an algebra with involution over K. Then the hyperbolic elements form a homogeneous ideal in $\widetilde{GW}(A,\sigma)$.*

Furthermore, if $f:(B,\tau)\to(A,\sigma)$ is a morphism in $\mathbf{Br}_h(K)$, then f_* sends hyperbolic elements of $\widetilde{GW}(B,\tau)$ to hyperbolic elements of $\widetilde{GW}(A,\sigma)$.

Proof. We start by proving the second statement. We show that for any morphisms f and g in $\mathbf{Br}_h(K)$, if g is hyperbolic and $f\circ g$ exists, then it is hyperbolic. If g is represented by (V,h) and f by (W,h'), then by hypothesis we have $U\subset V$ such that $U^\perp = U$. Now it is easy to see that $U\otimes W \subset V\otimes W$ is a Lagrangian for the form $h'\circ h$.

For the first statement, it is clear by definition that a sum of hyperbolic elements is hyperbolic, and that hyperbolicity may be checked component-wise. Let, for $=1,2$, (V_i,h_i) be a ε_i-hermitian module over (A_i,σ_i) where (A_i,σ_i) is either (A,σ) or (K,Id). Then if $U\subset V_1$ is a Lagrangian, it is easy to see that $U\otimes_K V_2$ is a Lagrangian in $(V_1\otimes_K V_2, h_1\otimes h_2)$, which is then hyperbolic. Now according to the point proved above, the product $(V_1, h_1) \cdot (V_2, h_2)$ in $\widetilde{GW}(A, \sigma)$ is hyperbolic since it is obtained from the tensor product by composition with some morphism in $\mathbf{Br}_h(K)$. $\square$

Me may now define:

Definition 2.21. *Let (A, σ) be an algebra with involution of over K. Then the mixed Witt ring $\widetilde{W}(A, \sigma)$ of (A, σ) is the quotient of $\widetilde{GW}(A, \sigma)$ by the ideal of hyperbolic elements.*

It is a commutative Γ-graded ring, and we write its homogeneous decomposition as

$$\begin{aligned}\widetilde{W}(A, \sigma) &= W(K) \oplus W(A, \sigma) \oplus W^-(A, \sigma)\\ &= W(K) \oplus W_+(A, \sigma) \oplus W_-(A, \sigma).\end{aligned}$$

Note that the component corresponding to $(0, 1) \in \Gamma$ is trivial, since all anti-symmetric forms over K are hyperbolic (in other words $W^-(K) = 0$).

From proposition 2.20 and theorem 2.18, we immediately deduce:

Theorem 2.22. *The functor $\widetilde{GW}$ naturally induces to a functor*

$$\widetilde{W} : \mathbf{Br}_h(K) \longrightarrow \mathbf{ComRing}_\Gamma$$

where $\mathbf{ComRing}_\Gamma$ is the category of Γ-graded commutative rings.

As before, we also have functors defined by $\widetilde{W}_\varepsilon(A, \sigma) = W(K) \oplus W_\varepsilon(A, \sigma)$.

2.5 Scalar extension and reciprocity

Scalar extension is a standard tool in the theory of algebras with involution, in particular when extending the scalars to a splitting field to reduce to the classical theory of bilinear forms over fields.

Proposition 2.23. *Let L/K be any field extension. For any algebra with involution (A, σ) over K, the canonical functor $\mathbf{Br}_h(K) \to \mathbf{Br}_h(L)$ described in proposition 1.20 induces graded (semi-)ring morphisms $\widetilde{SW}(A, \sigma) \to \widetilde{SW}(A_L, \sigma_L)$, $\widetilde{GW}(A, \sigma) \to \widetilde{GW}(A_L, \sigma_L)$ and $\widetilde{W}(A, \sigma) \to \widetilde{W}(A_L, \sigma_L)$.*

Proof. For $\widetilde{SW}$, the operations are preserved since direct sums and tensor products are compatible with scalar extension, and we noted in proposition 1.20 that $\varphi^{(d)}_{(A,\sigma)}$ was also preserved, which shows that the product in $\widetilde{SW}(A, \sigma)$ is compatible with scalar extension. For the grading, we just have to notice that the type of an involution is also stable under scalar extension.

The result then follows for $\widetilde{GW}$ by universal property, and for $\widetilde{W}$ since hyperbolic forms are clearly stable under base change. $\square$

Recall that an étale K-algebra is a finite product of finite separable field extensions of K. If L/K is an étale algebra (for instance a separable field extension), then it is known ([10, I.7.2,I.7.3.2]) that the trace map $\mathrm{Tr}_{L/K} : L \to K$ induces a group morphism

$$\begin{array}{rccc}(\mathrm{Tr}_{L/K})_* : & GW(L) & \longrightarrow & GW(K)\\ & (V, b) & \longmapsto & (V, \mathrm{Tr}_{L/K} \circ b)\end{array} \qquad (10)$$

which sends $\langle 1\rangle_L$ to the so-called *trace form* $T_{L/K} \in GW(K)$:

$$\begin{array}{rccc} T_{L/K}: & L\times L & \longrightarrow & K \\ & (x,y) & \longmapsto & \mathrm{Tr}_{L/K}(xy) \end{array}$$

(and the fact that this is a non-degenerated bilinear form is a characterization of étale algebras). The classical Frobenius reciprocity theorem (see [21, 2.5.6] for the case of a field extension) states that this is actually a $GW(K)$-module morphism, where $GW(L)$ is seen as a $GW(K)$-module through the natural map $GW(K) \to GW(L)$. It is easy to see that the trace map sends hyperbolic forms to hyperbolic forms, so it induces a morphism

$$(\mathrm{Tr}_{L/K})_* : W(L) \longrightarrow W(K) \tag{11}$$

which is a $W(K)$-module morphism. We want to generalize this result to mixed Witt rings.

Assume for simplicity that L/K is a finite separable field extension. Then the trace also gives a K-linear map

$$\mathrm{Tr}_{A_L/A} : A_L \xrightarrow{\mathrm{Id}_A \otimes \mathrm{Tr}_{L/K}} A. \tag{12}$$

We need some compatibility with the reduced trace:

Lemma 2.24. *Let $a,b \in A$ and $z \in (A\otimes_K A)_L$. Then*

$$\mathrm{Trd}_A(a(\mathrm{Tr}_{(A\otimes_K A)_L/A\otimes_K A}(z)\cdot b)) = \mathrm{Tr}_{L/K}(\mathrm{Trd}_{A_L}((a\otimes 1)(z\cdot b)))$$

where $A\otimes_K A$ acts on A through the twisted action induced by σ (see (2)).

Proof. By K-linearity, we may assume that $z = x\otimes y\otimes \lambda$ with $x,y\in A$ and $\lambda \in L$. The equation then becomes

$$\mathrm{Trd}_A(a\,\mathrm{Tr}_{L/K}(\lambda)xb\sigma(y)) = \mathrm{Tr}_{L/K}(\mathrm{Trd}_{A_L}(axb\sigma(y)\otimes\lambda))$$

which is clear since in general for any $e\in A$:

$$\mathrm{Tr}_{L/K}(\mathrm{Trd}_{A_L}(e\otimes\lambda)) = \mathrm{Tr}_{L/K}(\lambda)\,\mathrm{Trd}_A(e).$$ □

The linear map (12) induces a group morphism

$$\begin{array}{rccc} (\mathrm{Tr}_{A_L/A})_*: & GW^\varepsilon(A_L,\sigma_L) & \longrightarrow & GW^\varepsilon(A,\sigma) \\ & (V,h) & \longmapsto & (V, \mathrm{Tr}_{A_L/A}\circ b). \end{array} \tag{13}$$

It is also easy to see that it sends hyperbolic forms to hyperbolic forms, so it defines graded group morphisms

$$(\mathrm{Tr}_{A_L/A})_* : \widetilde{GW}(A_L,\sigma_L) \longrightarrow \widetilde{GW}(A,\sigma) \tag{14}$$

and

$$(\mathrm{Tr}_{A_L/A})_* : \widetilde{W}(A_L,\sigma_L) \longrightarrow \widetilde{W}(A,\sigma). \tag{15}$$

Proposition 2.25 (Frobenius reciprocity). *If we see $\widetilde{GW}(A_L,\sigma_L)$ as a $\widetilde{GW}(A,\sigma)$-module through the scalar extension map, then the trace map (14) is a $\widetilde{GW}(A,\sigma)$-module morphism. Likewise, the trace map (15) is a $\widetilde{W}(A,\sigma)$-module morphism.*

Proof. First, note that clearly it is enough to prove this for $\widetilde{GW}$, the case of $\widetilde{W}$ being an easy consequence, using that hyperbolic forms are preserved by the trace map.

We have to check that for $x \in \widetilde{GW}(A,\sigma)$, $y \in \widetilde{GW}(A_L,\sigma_L)$, we have

$$(\mathrm{Tr}_{A_L/A})_*(x_L \cdot y) = x \cdot (\mathrm{Tr}_{A_L/A})_*(y).$$

Of course it is only necessary to check this when x and y are homogeneous and represent ε-hermitian forms. So take (A_1,σ_1) and (A_2,σ_2) to each be either (A,σ) or (K,Id), and write (A_3,σ_3) for either (A,σ) or (K,Id), so that we have a canonical isomorphism

$$(A_1,\sigma_1)\otimes_K (A_2,\sigma_2) \xrightarrow{f} (A_3,\sigma_3)$$

in $\mathbf{Br}_h(K)$ given either by the fact that (K,Id) is the unit element of $\mathbf{Br}_h(K)$, or by $f = \varphi^{(2)}_{(A,\sigma)}$ (when $A_1 = A_2 = A$). Then to conclude we need to show that the following outer diagram commutes:

$$\begin{array}{ccc}
 & GW^{\varepsilon_1}(A_1,\sigma_1)\otimes GW^{\varepsilon_2}((A_2,\sigma_2)_L) & \\
 \swarrow^{ext_L\otimes 1} & & \searrow^{1\otimes(\mathrm{Tr}_{(A_2)_L/A_2})_*} \\
GW^{\varepsilon_1}((A_1,\sigma_1)_L)\otimes GW^{\varepsilon_2}((A_2,\sigma_2)_L) & & GW^{\varepsilon_1}(A_1,\sigma_1)\otimes GW^{\varepsilon_2}(A_2,\sigma_2) \\
\downarrow & & \downarrow \\
GW^{\varepsilon_1\varepsilon_2}((A_1\otimes_K A_2,\sigma_1\otimes\sigma_2)_L) & \xrightarrow{(\mathrm{Tr}_{(A_1\otimes_K A_2)_L/A_1\otimes_K A_2})_*} & GW^{\varepsilon_1\varepsilon_2}(A_1\otimes_K A_2,\sigma_1\otimes\sigma_2) \\
\downarrow{\scriptstyle (f_L)_*} & & \downarrow{\scriptstyle f_*} \\
GW^{\varepsilon_1\varepsilon_2}((A_3,\sigma_3)_L) & \xrightarrow{(\mathrm{Tr}_{(A_3)_L/A_3})_*} & GW^{\varepsilon_1\varepsilon_2}(A_3,\sigma_3).
\end{array}$$

Independently of the particular values of A_1 and A_2, it is not difficult to see that the top pentagon commutes, extending the proof of the classical case (where $A_1 = A_2 = K$). Indeed, take (V_1,h_1) a ε_1-hermitian module over (A_1,σ_1), and (V_2,h_2) a ε_2-hermitian module over $(A_2,\sigma_2)_L$. Then the natural map

$$\begin{array}{rcl}
(V_1\otimes_K L)\otimes_L V_2 & \longrightarrow & V_1\otimes_K V_2 \\
(x\otimes\lambda)\otimes y & \longmapsto & x\otimes(\lambda y)
\end{array}$$

is an module isomorphism over

$$(A_1\otimes_K L)\otimes_L (A_2\otimes_K L) \simeq A_1\otimes_K A_2\otimes_K L,$$

and we see that it is an isometry between $(\mathrm{Id}_{A_L}\otimes \mathrm{Tr}_{A_L/A})\circ((h_1\otimes 1)\otimes h_2)$ and $h_1\otimes(\mathrm{Tr}_{A_L/A}\circ h_2)$ under this identification.

When A_1 or A_2 is K, the bottom square is clearly commutative by the usual compatibilities of tensoring with K. It remains to show that is commutes when $A_1 = A_2 = A$, so $A_3 = K$ and $f = (A, T_\sigma)$, which gives:

$$\begin{array}{ccc}
GW^{\varepsilon}((A\otimes_K A,\sigma\otimes\sigma)_L) & \xrightarrow{(\mathrm{Tr}_{(A\otimes_K A)_L/A\otimes_K A})_*} & GW^{\varepsilon}(A\otimes_K A,\sigma\otimes\sigma) \\
\downarrow{\scriptstyle (T_{\sigma_L})_*} & & \downarrow{\scriptstyle (T_\sigma)_*} \\
GW^{\varepsilon}(L) & \xrightarrow{(\mathrm{Tr}_{L/K})_*} & GW^{\varepsilon}(K).
\end{array}$$

Let (V, h) be a ε-hermitian module over $(A \otimes_K A, \sigma \otimes \sigma)_L$. Then we have to find an isometry between the K-bilinear spaces (U, β) and (W, β') where $U = V \otimes_{A \otimes_K A} A$ and $W = V \otimes_{A_L \otimes_L A_L} A_L$, with

$$\beta(u \otimes a, v \otimes b) = \mathrm{Trd}_A \left(\sigma(a)(\mathrm{Tr}_{(A \otimes_K A)_L / A \otimes_K A}(h(u, v)) \cdot b)\right)$$

and

$$\beta'(u \otimes a \otimes \lambda, v \otimes b \otimes \mu) = \mathrm{Tr}_{L/K} \left(\mathrm{Trd}_{A_L}((\sigma(a) \otimes \lambda)(h(u, v) \cdot (b \otimes \mu)))\right).$$

Now if we consider the following K-linear map

$$\begin{array}{rccc} \Phi : & V \otimes_{A \otimes_K A} A & \longrightarrow & V \otimes_{A_L \otimes_L A_L} A_L \\ & v \otimes a & \longmapsto & v \otimes (a \otimes 1), \end{array}$$

then $\beta'(\Phi(u \otimes a), \Phi(v \otimes b)) = \beta(u \otimes a, v \otimes b)$ is a direct application of lemma 2.24 with $z = h(u, v)$. □

Remark 2.26. In particular, the image of the trace map is an ideal in $\widetilde{GW}(A, \sigma)$ (resp. $\widetilde{W}(A, \sigma)$), which as in the classical case we call the *trace ideal* (relative to L/K).

2.6 The split case

The rings we defined have an interesting description when $(A, \sigma) = (K, \mathrm{Id})$, which through Morita equivalence is useful whenever A is split. Indeed, we have by construction

$$\widetilde{GW}(K, \mathrm{Id}) = GW^{\pm}(K) \oplus GW^{\pm}(K)$$
$$\widetilde{W}(K, \mathrm{Id}) = W(K) \oplus W(K),$$

so the even and the odd components are identical. Furthermore, since the rings are Γ-graded, the even components are naturally $\mathbb{Z}/2\mathbb{Z}$-graded rings (it is a little less natural for the case of $\widetilde{W}(K, \mathrm{Id})$ since the component $W^-(K)$ is trivial).

Proposition 2.27. *The ring $\widetilde{GW}(K, \mathrm{Id})$ (resp. $\widetilde{W}(K, \mathrm{Id})$) is canonically isomorphic to the group ring $GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$ (resp. $W(K)[\mathbb{Z}/2\mathbb{Z}]$) as Γ-graded rings.*

Proof. This follows easily from the definition of the product in $\widetilde{GW}(K, \mathrm{Id})$ and the fact that $\varphi^{(d)}_{(K,\mathrm{Id})}$ is essentially the identity of (K, Id) in $\mathbf{Br}_h(K)$, so that even for the orthogonal and symplectic components of $\widetilde{GW}(K, \mathrm{Id})$, the product is simply given by the tensor product, as in $GW^{\pm}(K)$. □

2.7 Twisted involution trace forms

Now that we have established the formal properties of our rings, we would like to be able to perform explicit computations. Obviously we can compute any product if we can describe products of homogeneous elements in $\widetilde{SW}(A, \sigma)$, and a quick examination of the cases shows that only products $xy \in SW(K)$ with $x, y \in SW^{\varepsilon}(A, \sigma)$ are difficult to compute. It is natural to specialize to the case where x and y are diagonal, and by bilinearity we just need to describe a product $\langle a \rangle_\sigma \langle b \rangle_\sigma$.

Proposition 2.28. *Let (A,σ) be an algebra with involution over K. Let a and b be ε-symmetric invertible elements of A. Then in $\widetilde{SW}(A,\sigma)$ we have:*

$$\langle a\rangle_\sigma\langle b\rangle_\sigma = T_{\sigma,a,b}$$

where $T_{\sigma,a,b}$ is the symmetric bilinear form given by

$$\begin{array}{rcl} A\times A & \longrightarrow & K \\ (x,y) & \longmapsto & \mathrm{Trd}_A(\sigma(x)ay\sigma(b)). \end{array}$$

In particular, $\langle 1\rangle_\sigma\langle a\rangle_\sigma = T_{\sigma,a}$, where $T_{\sigma,a} = T_{\sigma,1,a}$ is the so-called twisted involution trace form (see [11, §11]).

Proof. From remarks 2.5 and 2.6 we have $\langle a\rangle_\sigma\langle b\rangle_\sigma$ defined on $(A\otimes_K A)\otimes_{A\otimes_K A} A$, and it sends $(x\otimes y\otimes z, x'\otimes y'\otimes z')$ to $\mathrm{Trd}_A\left(\sigma(z)(\sigma(x)ax')z'\sigma(\sigma(y)by')\right)$. If we identify $(A\otimes_K A)\otimes_{A\otimes_K A} A$ to A by $a\mapsto (1\otimes 1)\otimes a$, then we find the expected formula, taking $x=x'=y=y'=1$. □

Example 2.29. In particular, $\langle 1\rangle_\sigma^2 = T_\sigma$, which of course follows directly from the definition of the product. The idea that T_σ represents in some sense the "square" of the involution σ has appeared in the literature in various forms, for instance in the definition of the signature of an involution (we explore this in more details in section 6). Our construction gives some solid ground to this idea.

Remark 2.30. Proposition 2.28 can theoretically be used to compute any product of ε-hermitian forms: if $h,h'\in SW^\varepsilon(A,\sigma)$, choose any involution θ on the division algebra D in the Brauer class of A, and choose some hermitian form f over (D,θ) such that $\sigma=\sigma_f$. Then $f_*(h) = \langle a_1,\dots,a_n\rangle_\theta$, and $f_*(h') = \langle b_1,\dots,b_m\rangle_\theta$, so $h\cdot h' = \sum_{i,j} T_{\theta,a_i,b_j}$ (and of course this quadratic form is independent of the choice of f).

Corollary 2.31. *Let (A,σ) be an algebra with involution over K, and let (V,h) and (V,h') be ε-hermitian forms over (A,σ). Setting $B=\mathrm{End}_A(V)$ and $\tau=\sigma_h$, there is a unique $a\in B^*$ such that, for all $x,y\in V$, $h'(x,y)=h(x,ay)$. Then $\tau(a)=a$ and*

$$h\cdot h' = T_{\tau,a}.$$

In particular, $h^2 = T_\tau$.

Proof. The element $a\in B$ is by definition the A-morphism $V\to V$ given by $\hat{h}^{-1}\circ\hat{h}'$, which proves its existence and unicity. The fact that h' is ε-hermitian implies that $\tau(a)=a$: indeed, we have $h'(y,x)=\varepsilon\sigma(h'(x,y)) = h(ay,x)$ but also $h'(y,x)=h(y,ax)=h(\tau(a)y,x)$, and this since this true for all x and y, $\tau(a)=a$.

Let $f:(B,\tau)\to(A,\sigma)$ be the morphism in $\mathbf{Br}_h(K)$ corresponding to (V,h). Then $f_*(\langle 1\rangle_\tau) = (V,h)$, and $f_*(\langle a\rangle_\tau)=h'$ (see example 2.8). Thus since f_* is a ring morphism we find $h\cdot h' = \langle 1\rangle_\tau\cdot\langle a\rangle_\tau = T_{\tau,a}$. □

This means that we can reinterpret twisted involution forms as being exactly the products of ε-hermitian forms having the same dimension. These computations show that understanding the product in $\widetilde{GW}(A,\sigma)$ and $\widetilde{W}(A,\sigma)$ amounts to understanding twisted involution trace forms (usually for involutions different from σ). In section 3 we give some examples of explicit computations in some cases involving crossed products, notably for quaternion algebras.

2.8 The dimension map

There is an obvious map $\dim : SW(K) \to \mathbb{N}$ that sends a quadratic space to its dimension, which extends to a ring morphism $GW(K) \to \mathbb{Z}$. It naturally induces a commutative diagram of rings

$$\begin{array}{ccc} GW(K) & \xrightarrow{\dim} & \mathbb{Z} \\ \downarrow & & \downarrow \\ W(K) & \xrightarrow{\dim_2} & \mathbb{Z}/2\mathbb{Z}. \end{array}$$

The morphism $\dim_2 : W(K) \to \mathbb{Z}/2\mathbb{Z}$ has a special importance in the theory of quadratic forms, mainly through its kernel $I(K)$, which is called the fundamental ideal of $W(K)$. The filtration $I^n(K)$ of $W(K)$ is in turn called the fundamental filtration. Amongst the most striking results in quadratic form theory, the Arason-Pfister Hauptsatz implies that the fundamental filtration is separated (meaning $\bigcap_n I^n(K) = 0$, see [12, X.5.2]), and the Milnor conjecture (proved by Voevodsky in [22]) states that the associated graded ring is the cohomology ring $H^*(K, \mathbb{Z}/2\mathbb{Z})$. To alleviate the notations, for the rest of this section we will write $H^*(K)$ for $H^*(K, \mathbb{Z}/2\mathbb{Z})$.

Let (A, σ) be an algebra with involution over K. Then the map $SW(K) \to \mathbb{N}$ extends to

$$\widetilde{\mathrm{rdim}} : \widetilde{SW}(A, \sigma) \longrightarrow \mathbb{N}[\Gamma]$$

called the (graded) *dimension map*, sending a ε-hermitian module to its reduced dimension (with the appropriate grading).

Proposition 2.32. *Let (A, σ) be an algebra with involution over K. The dimension map is a Γ-graded semi-ring morphism, that induces a commutative diagram of Γ-graded rings*

$$\begin{array}{ccc} \widetilde{GW}(A, \sigma) & \xrightarrow{\widetilde{\mathrm{rdim}}} & \mathbb{Z}[\Gamma] \\ \downarrow & & \downarrow \\ \widetilde{W}(A, \sigma) & \xrightarrow{\widetilde{\mathrm{rdim}}_2} & \mathbb{Z}/2\mathbb{Z}[\Gamma]. \end{array}$$

Proof. By definition, the dimension map sends homogeneous component to homogeneous component. Since the reduced dimension of a direct sum (resp. a tensor product) of modules is the sum (resp. product) of the reduced dimensions, it defines a semi-ring morphism $\widetilde{SW}(A, \sigma) \to \mathbb{N}[\Gamma]$, which by universal property extends uniquely to a (graded) ring morphism $\widetilde{GW}(A, \sigma) \to \mathbb{Z}[\Gamma]$.

Since hyperbolic modules have even reduced dimension, the map $\widetilde{GW}(A, \sigma) \to \mathbb{Z}/2\mathbb{Z}[\Gamma]$ factors through $\widetilde{W}(A, \sigma)$. □

If we compose with the augmentation map $\mathbb{Z}[\Gamma] \to \mathbb{Z}$, we get a ring morphism $\mathrm{rdim} : \widetilde{GW}(A, \sigma) \to \mathbb{Z}$ we call the *total dimension map*. It induces a commutative diagram of rings:

$$\begin{array}{ccccc} GW(K) & \longrightarrow & \widetilde{GW}(A,\sigma) & \xrightarrow{\mathrm{rdim}} & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ W(K) & \longrightarrow & \widetilde{W}(A,\sigma) & \xrightarrow{\mathrm{rdim}_2} & \mathbb{Z}/2\mathbb{Z}. \end{array}$$

3 Crossed products and twisted trace forms

We saw in corollary 2.31 that computing products in $\widetilde{W}(A,\sigma)$ amounts to computing twisted involution trace forms (potentially for other algebras with involution). Although they have a very simple definition, these quadratic forms are usually difficult to actually compute. In this section, we give some results in very special cases, namely we give an explicit description of $T_{\sigma,a}$ when (A,σ) is a crossed product with involution and a is one of the generators of the crossed product (proposition 3.12).

3.1 Galois algebras

Our notion of crossed product is slightly more general than the one often found in literature (but less general than some other versions), in the sense that we allow G-Galois algebras instead of only Galois field extensions. This has the advantage to be compatible with separable base change, and it would be necessary if we wanted to consider unitary involutions (though we limit ourselves to involutions of the first kind here to stay coherent with the rest of the article). We give a brief review on these algebras, based on [11, 18.B].

Let G be a finite group. Then recall that a G-Galois algebra over K is an étale K-algebra (thus a finite product of finite separable field extensions of K) endowed with an action of G by K-automorphisms, such that $L^G = K$ and $|G| = \dim_K(L)$.

Example 3.1. Clearly, any Galois field extension L/K is a $\mathrm{Gal}(L/K)$-Galois algebra, and any G-Galois algebra structure on L is given by some isomorphism $G \simeq \mathrm{Gal}(L/K)$.

Any G-Galois algebra can be constructed from a Galois field extension in an elementary way.

Definition 3.2. *Let M be any K-algebra, and let H be a group acting on M by K-automorphisms. Then if $H \subset G$, we define the induced algebra*

$$\mathrm{Ind}_H^G(M) = \mathrm{Hom}_H(G, M)$$

where G and M are naturally viewed as left H-sets. Its K-algebra structure is given pointwise by that of M, and we have a left action of G on $\mathrm{Ind}_H^G(M)$ induced by the right action of G on itself: $(g \cdot f)(x) = f(xg)$.

Note that as a K-algebra, we have $\mathrm{Ind}_H^G(M) \approx M^{[G:H]}$, though the isomorphism is not canonical and requires a choice of representatives of classes in G/H. Then we can state:

Proposition 3.3. *Let M/K be a H-Galois algebra, and let G be a finite group with $H \subset G$. Then $\mathrm{Ind}_H^G(M)$ is a G-Galois algebra, isomorphic as a K-algebra to $M^{[G:H]}$.*

Furthermore, let L/K be a G-Galois algebra. Then there exists a unique $H \subset G$ and a unique H-Galois field extension M/K such that $L = \mathrm{Ind}_H^G(M)$.

Proof. This is a paraphrase of propositions [11, 18.17, 18.18]. □

Remark 3.4. In particular, in general a given étale algebra L can be a G-Galois algebra for non-isomorphic groups G. Indeed, L has a structure of G-Galois algebra for *some* G iff as an alegbra it has the form $L \simeq M^n$ for some Galois extension M/K, and it has a G-Galois algebra structure for any group G that has $\mathrm{Gal}(M/K)$ as an index n subgroup.

3.2 Crossed products

We can now define our notion of crossed product:

Definition 3.5. *Let A be a central simple algebra over K, let G be a finite group with $|G| = \deg(A)$, and let L be a G-Galois algebra over K. We say that A is a (G, L)-crossed product if it is equipped with a K-embedding $L \to A$. An isomorphism of (G, L)-crossed product is an isomorphism of algebras which commutes with the embeddings, and we write $X(G, L)$ for the set of isomorphism classes of (G, L)-crossed products.*

Remark 3.6. For division algebras this is equivalent to the more usual definition using Galois field extension, but for general central simple algebras this definition is strictly larger, since for instance over a separably closed field K, the matrix algebra $M_2(K)$ is a $(\mathbb{Z}/2\mathbb{Z}, K^2)$-crossed product with this definition, but cannot have an embedded quadratic subfield.

We have the usual presentation for crossed products, identical to the classical case where L is only allowed to be a field. This is justified by the following variant of the Skolem-Noether theorem:

Proposition 3.7. *Let A be a central simple algebra over K, and let L be an étale K-algebra such that $[L : K] = \deg(A)$. Then for any two K-algebra embeddings $j_1, j_2 : L \to A$, there exists $a \in A^*$ such that $j_2(x) = aj_1(x)a^{-1}$ for all $x \in L$. Furthermore, if we identify L with a subalgebra of A, $C_A(L) = L$, where $C_A(L)$ is the centralizer of L in A.*

Proof. Assume L is a subalgebra of A, through either one of the embeddings. We have $L \subset C_A(L)$ since L is commutative, and we can check for instance after splitting A that $L = C_A(L)$ (since the elements of L are simultaneously diagonalizable).

Note that since A_L is a finite direct product of central simple algebras over fields, if two A_L-modules have isomorphic endomorphism algebras, they are isomorphic as module. We define two structures of right A_L-module on A, by $x \cdot_i (a \otimes \lambda) = j_i(\lambda)xa$ for $i = 1, 2$, and we write A_i for the corresponding module. Then $\mathrm{End}_{A_L}(A_i) = C_A(j_i(L)) = j_i(L) \simeq L$, so $A_1 \simeq A_2$ as A_L-modules. If $f : A_1 \to A_2$ is a module isomorphism, then a simple calculation shows that $a = f(1)$ satisfies $j_2(x) = aj_1(x)a^{-1}$ for all $x \in L$. □

Thus if A is a (G,L)-crossed product, where we identify L with a subalgebra of A, for any $g \in G$ we can set

$$\mathcal{T}_g(A) = \{a \in A \mid \forall \lambda \in L,\, a\lambda = g(\lambda)a\} \tag{16}$$

and the proposition implies that each $\mathcal{T}_g(A)$ is a L-line containing invertible elements. As K-vector spaces, we have

$$A = \bigoplus_{g\in G} \mathcal{T}_g(A).$$

Note that $\mathcal{T}_1(A) = L$, that $\mathcal{T}_g(A)\mathcal{T}_h(A) = \mathcal{T}_{gh}(A)$, and that Trd_A is zero on $\mathcal{T}_g(A)$ if $g \neq 1$. We say that a family $(u_g)_{g\in G}$ is a choice of *standard generators* of A if $u_g \in \mathcal{T}_g(A)$ is an invertible element for all $g \in G$.

Then we easily see that for any $g, h \in G$, we have

$$u_g u_h = \alpha(g,h) u_{gh} \tag{17}$$

for some $\alpha(g,h) \in L^*$, and the associativity of A shows that $(g,h) \mapsto \alpha(g,h)$ is a 2-cocycle in $Z^2(G, L^*)$. Furthermore, if we choose different generators $u'_g = c_g u_g$ for some cochain $(c_g) \in C^1(G, L^*)$, we get

$$\alpha'(g,h) = \frac{c_g g(c_h)}{c_{gh}} \alpha(g,h) \tag{18}$$

so the cohomology class $[\alpha] \in H^2(G, L^*)$ is well-defined. This defines an injective map

$$X(G,L) \longrightarrow H^2(G, L^*).$$

Remark 3.8. At least for $g = 1$, there is a canonical choice for u_1, namely the unit 1 of A. Making this choice amounts to having α be a *reduced* cocycle, meaning that $\alpha(g,h) = 1$ whenever g or h is 1. Unless otherwise specified, we will always make this choice.

Conversely, given any 2-cocycle $\alpha \in Z^2(G, L^*)$ where L is a G-Galois algebra, we can define a central simple algebra A_α generated by L and some elements u_g satisfying the relations (16) and (17). This means that the map $X(G,L) \to H^2(G,L^*)$ described above is actually bijective. Note that there is an obvious surjective map $X(G,L) \to \mathrm{Br}(L/K)$, which is well-known to be injective, meaning that if there is an algebra isomorphism between two (G,L)-crossed product, then there is an isomorphism that preserves the embedding of L. This is one way to describe the classical isomorphism $\mathrm{Br}(L/K) \simeq H^2(G, L^*)$.

3.3 Crossed products with involution

Now we examine what happens if we add an involution on such a crossed product.

Definition 3.9. *Let G be a finite group, L be a G-Galois algebra over K, and $\sigma \in Aut_K(L)$ be of order at most 2 such that $\sigma G \sigma = G$ (where we see G as a subgroup of $\mathrm{Aut}_K(L)$).*

Let (A,θ) be an algebra with involution over K. We say that (A,θ) is a (G,L,σ)-crossed product if it is a (G,L)-crossed product such that $\theta_{|L} = \sigma$

(where we see L as a subalgebra of A). We then usually write $\theta = \sigma$. An isomorphism of (G, L, σ)-crossed product is an isomorphism of (G, L)-crossed product which is also an isomorphism of algebras with involution. We write $X(G, L, \sigma)$ for the set of isomorphism classes of (G, L, σ)-crossed products.

Remark 3.10. The condition $\sigma G \sigma = G$ is in particular satisfied when $\sigma \in G$. This is always the case when L is a field, since σ is a K-automorphism of L, hence an element of $\mathrm{Gal}(L/K)$. In [9], only the case where L is a field is considered, but they also allow unitary involutions, in which case obviously $\sigma \notin G$. Our formalism can be easily adapted to unitary involutions as well (we just need to allow σ to be in $\mathrm{Aut}_k(L)$ where K/k is a quadratic extension).

We now explain how to describe $X(G, L, \sigma)$ in a similar way to how we described $X(G, L)$. Note that there is an obvious map $X(G, L, \sigma) \to X(G, L)$.

For any $g \in G$, we set

$$\overline{g} = \sigma g^{-1} \sigma \in G$$

and

$$\sigma_g = g\sigma \in \mathrm{Aut}_K(L).$$

Then $g \mapsto \overline{g}$ is an anti-automorphism of order at most 2 of G. Furthermore, σ_g is involutive iff $g = \overline{g}$, which happens in particular when $g = 1$ (in which case $\sigma_g = \sigma$) and $g = \sigma$ (when $\sigma \in G$, in which case $\sigma_g = \mathrm{Id}_L$).

We define $Z(G, L, \sigma) \subset Z^2(G, L^*) \times C^1(G, L^*)$ such that $(\alpha, (\mu_g))$ is in $Z(G, L, \sigma)$ iff for all $g, h \in G$:

$$\mu_{\overline{g}} \sigma_g(\mu_g) = 1 \tag{19}$$

and

$$\mu_h \overline{g}(\mu_g) \alpha(\overline{h}, \overline{g}) = \mu_{gh} \sigma_{\overline{gh}}(\alpha(g, h)). \tag{20}$$

We define a group morphism:

$$\begin{array}{rccl} \delta: & C^1(G, L^*) & \longrightarrow & Z(G, L, \sigma) \\ & (c_g) & \longmapsto & \left((g, h) \mapsto \frac{c_g g(c_h)}{c_{gh}}, \left(\frac{\sigma_{\overline{g}}(c_g)}{c_{\overline{g}}} \right) \right). \end{array}$$

The fact that it is well-defined and a group morphism results from a simple computation. Then we set

$$H(G, L, \sigma) = \mathrm{Coker}(\delta).$$

Proposition 3.11. *Let G, L and σ be as in definition 3.9, and let (A, σ) be a (G, L, σ)-crossed product. Then for any $g \in G$ we have $\sigma(\mathcal{T}(A)) = \mathcal{T}_{\overline{g}}(A)$. In particular, for any choice of standard generators u_g, there is a unique family $(\mu_g)_{g \in G} \in C^1(G, L^*)$ such that*

$$\sigma(u_g) = \mu_g u_{\overline{g}} \tag{21}$$

for all $g \in G$.

Then if $\alpha \in Z^2(G, L^*)$ is the 2-cocycle associated to the generators (u_g), we have $(\alpha, (\mu_g)) \in Z(G, L, \sigma)$. Any other choice $u'_g = c_g u_g$ of generators leads to the element

$$(\alpha', (\mu'_g)) = \delta((c_g)) \cdot (\alpha, (\mu_g))$$

so that we have a well-defined map

$$X(G, L, \sigma) \longrightarrow H(G, L, \sigma).$$

Furthermore, this map is a bijection that makes the following diagram commute:

$$\begin{array}{ccc} X(G,L,\sigma) & \longrightarrow & H(G,L,\sigma) \\ \downarrow & & \downarrow \\ X(G,L) & \longrightarrow & H^2(G,L^*). \end{array}$$

Proof. Let $g \in G$, $a \in \mathcal{T}_g(A)$ and $\lambda \in L$. Then applying σ to the equation $a(g^{-1}\sigma)(\lambda) = \sigma(\lambda)a$ we get

$$\sigma(a)\lambda = \overline{g}(\lambda)\sigma(a),$$

so $\sigma(a) \in \mathcal{T}_{\overline{g}}(A)$. The uniqueness of $\mu_g \in L^*$ such that equation (21) holds is justified by the invertibility of u_g. The equations (19) and (20) are immediately obtained from $\sigma^2(u_g) = u_g$ and $\sigma(u_g u_h) = \sigma(u_h)\sigma(u_g)$.

If $u'_g = c_g u_g$, then α' is given by the usual cocycle equation (18), and

$$\sigma(u'_g) = \sigma(c_g u_g) = \mu_g u_{\overline{g}} \sigma(c_g) = \mu_g \frac{\overline{g}\sigma(c_g)}{c_{\overline{g}}} u'_{\overline{g}},$$

which by definition of the map δ is the expected relation.

The diagram and its commutativity are obvious: the map from $H(G, L, \sigma)$ to $H^2(G, L^*)$ is induced by the natural projection from $Z(G, L, \sigma)$ to $Z^2(G, L^*)$, and by definition this commutes with the coboundary maps from $C^1(G, L^*)$. The only left to check is the fact that $X(G, L, \sigma) \to H(G, L, \sigma)$ is bijective. For injectivity, suppose (A', σ) is another (G, L, σ)-crossed product defining the same class as (A, σ) in $H(G, L, \sigma)$. Then we can choose standard generators in A' which define the same element in $Z(G, L, \sigma)$; but these relations completely characterize the algebra structure of A' and the action of σ, so actually (A, σ) and (A', σ) are isomorphic as (G, L, σ)-crossed products. For surjectivity, given any element $(\alpha, (\mu_g)) \in Z(G, L, \sigma)$, we can define A from α as for usual (G, L)-crossed products, and define the involution σ on A by having it act as $\sigma \in \mathrm{Aut}_K(L)$ on the copy of L in A, and on the u_g by equation (21). Then condition (20) will ensure that it is an anti-morphism, and condition (19) will impose that it is involutive. □

3.4 Twisted trace forms

Now that we gave a way to describe crossed products with involution, we can compute certain twisted involution trace forms. Precisely, let (A, σ) be a (G, L, σ)-crossed product over K. Then we want to describe $T_{\sigma,a}$ where $a \in \mathcal{T}_t(A)$ is an invertible element in $\mathrm{Sym}(A, \sigma)$, for some $t \in G$. According to proposition 3.11, this imposes that $\bar{t} = t$ (see section 3.5 for a discussion on the existence of such an element a).

Thus we set $S = \{t \in G \,|\, \bar{t} = t\}$. Note that $1 \in S$ and $\sigma \in S$ whenever $\sigma \in G$ (which happens for instance when L is a field). When $\sigma = 1$, then S is the set of elements of order at most 2 in G, and when $\sigma \neq 1$ then $t \in S$ iff σ and t generate a dihedral subgroup of $\mathrm{Aut}_K(L)$.

We also set for any $t \in S$:

$$G_t = \{g \in G \mid [\sigma, g] = t\}$$

where $[g,h] = ghg^{-1}h^{-1}$ is the classical commutator in G (note that $[\sigma, g]$ is always in S). In particular, G_1 is the centralizer $C_G(\sigma)$ of σ in G, and if $\sigma \in G$ then $G_\sigma = \emptyset$. When $\sigma = 1$, then $G_1 = G$ and $G_t = \emptyset$ for any non-trivial $t \in S$.

Proposition 3.12. *Let (A, σ) be a (G, L, σ)-crossed product over K, and let $t \in S$. Suppose $a \in \mathcal{T}_t(A)$ is invertible and symmetric. For any choice of standard generators on A (with $a = \xi u_t$), we set for all $g \in G_t$:*

$$\omega_g = \sigma(u_g) a u_g = \overline{g}(\xi)\mu_g \alpha(\overline{g}, t)\alpha(\overline{g}t, g) \in (L^\sigma)^*.$$

We then define $q_a \in W(L^\sigma)$ by, if $\sigma = 1$:

$$q_a = \sum_{g \in G_t} \langle \omega_g \rangle;$$

and if $\sigma \neq 1$ and $L = L^\sigma(\sqrt{d})$:

$$q_a = \langle 2 \rangle \langle\langle d \rangle\rangle \sum_{g \in G_t} \langle \omega_g \rangle.$$

Then q_a is independent of the choice of standard generators, and we have in $W(K)$:

$$T_{\sigma,a} = (\mathrm{Tr}_{L^\sigma/K})_*(q_a)$$

where $(\mathrm{Tr}_{L^\sigma/K})_* : W(L^\sigma) \to W(K)$ is as in (11).

Proof. For any $g \in G_t$, if we set $u'_g = c u_g$ for some $c \in L^*$, then we get

$$\begin{aligned} \omega'_g &= \sigma(c u_g) a (c u_g) \\ &= (\overline{g}\sigma)(c) \cdot (\overline{g}t)(c) \cdot \omega_g. \end{aligned}$$

Suppose first that $\sigma = 1$. Then if $t \neq 1$ we have $G_t = \emptyset$ so $q_a = 0$ is indeed independent of the choice of generators. If $t = 1$, then for any $g \in G$ we have $\omega'_g = (g^{-1}(c))^2 \omega_g$, so $\langle \omega'_g \rangle = \langle \omega_g \rangle$ in $GW(L)$, and q_a is again independent of the choice.

Now assume that $\sigma \neq 1$. Then we get

$$\omega'_g = N_{L/L^\sigma}(g^{-1}(c))\omega_g,$$

and for any $x \in N_{L/L^\sigma}(L^*)$ we have $\langle x \rangle \langle\langle d \rangle\rangle = \langle\langle d \rangle\rangle$ since $\langle\langle d \rangle\rangle$ is the norm form of L/L^σ (so x is represented by $\langle\langle d \rangle\rangle$, and thus it is a similarity factor). So in the end $\langle \omega'_g \rangle \langle\langle d \rangle\rangle = \langle \omega_g \rangle \langle\langle d \rangle\rangle$, and q_a is well-defined.

We define the function $\varphi : G \to G$ by $\overline{g}t\varphi(g) = 1$. Let $g, h \in G$ with $\varphi(g) \neq h$. Then for any $x \in \mathcal{T}_g(A)$ and $y \in \mathcal{T}_h(A)$, $\sigma(x) a y \in \mathcal{T}_{\overline{g}th}(A)$, so since by hypothesis $\overline{g}th \neq 1$, we have $\mathrm{Trd}_A(\sigma(x)ay) = 0$, and x and y are orthogonal for $T_{\sigma,a}$.

It is easy to see that $\varphi^2 = \mathrm{Id}_G$, so if we write Z for the set of orbits of φ, we have a natural decomposition $Z = Z_1 \cup Z_2$ between orbits of size 1 and 2, and $\{g\} \in Z_1$ iff $g \in G_t$. For any $z \in Z$, we write $V_z = \bigoplus_{g \in z} \mathcal{T}_g(A)$. Then what we just showed implies that the V_z are orthogonal for $T_{\sigma,a}$, and that if $z = \{g, h\}$ then $\mathcal{T}_g(A)$ is a Lagrangian for V_z, so V_z is a hyperbolic subspace when $z \in Z_2$.

For any $g \in G_t$, under the K-isomorphism

$$\begin{array}{ccc} L & \xrightarrow{\sim} & \mathcal{T}_g(A) \\ x & \longmapsto & g(x)u_g, \end{array}$$

the restriction of the bilinear form $T_{\sigma,a}$ to $\mathcal{T}_g(A)$ corresponds to

$$\begin{array}{cccc} b_g : & L \times L & \longrightarrow & K \\ & (x,y) & \longmapsto & \mathrm{Tr}_{L/K}(\sigma(x)y\omega_g), \end{array}$$

so in $W(K)$ we have $T_{\sigma,a} = \sum_{g\in G_t} b_g$. If $\sigma = 1$, then $b_g = (\mathrm{Tr}_{L/K})_*(\langle \omega_g \rangle)$; if $\sigma \neq 1$, then

$$\begin{aligned} b_g(x,y) &= \mathrm{Tr}_{L^\sigma/K}\left(\mathrm{Tr}_{L/L^\sigma}(\sigma(x)y\omega_g)\right) \\ &= \mathrm{Tr}_{L^\sigma/K}\left(\omega_g \,\mathrm{Tr}_{L/L^\sigma}(\sigma(x)y)\right). \end{aligned}$$

Now taking a L^σ-basis $(1, \sqrt{d})$ of L, we see that the form $(x,y) \mapsto \mathrm{Tr}_{L/L^\sigma}(\sigma(x)y)$ is isometric to $\langle 2\rangle \langle\langle d\rangle\rangle$, so $b_g = (\mathrm{Tr}_{L^\sigma/K})_*(\langle 2\rangle\langle\langle d\rangle\rangle\langle\omega_g\rangle)$. In any case, we showed that $T_{\sigma,a} = (\mathrm{Tr}_{L^\sigma/K})_*(q_a)$. □

Example 3.13. In particular, if $t = \sigma$, then $T_{\sigma,a}$ is hyperbolic, and if $\sigma = 1$, then $T_{\sigma,a}$ is hyperbolic when $t \neq 1$. Furthermore, taking $a = 1$, the Witt index of T_σ is at most $\deg(A) \cdot |C_G(\sigma)|$.

Corollary 3.14. *Let (A, σ) be a (G, L, σ)-crossed product over K, let $s, t \in S$, and let $x \in \mathcal{T}_s(A)$, $y \in \mathcal{T}_t(A)$ be invertible ε-symmetric elements (for some $\varepsilon = \pm 1$).*

For any choice of standard generators on A (with $x = \xi u_s$ and $y = \xi' u_t$), we set for all $g \in G$ such that $[\sigma s, g] = s^{-1}t$:

$$\begin{aligned} \omega_g &= \varepsilon x^{-1}\sigma(u_g)yu_g \\ &= \varepsilon s^{-1}\left(\frac{\overline{g}(\xi')}{\xi}\mu_g\right)\alpha(s^{-1},\overline{g})\alpha(s^{-1}\overline{g},t)\alpha(s^{-1}\overline{g}t,y) \in (L^{\sigma s})^*. \end{aligned}$$

We then define $q_{x,y} \in W(L^{\sigma s})$ by: if $s = \sigma$ and $t \neq \sigma$ then $q_{x,y} = 0$; if $s = t = \sigma$,

$$q_{x,y} = \sum_{g\in G} \langle \omega_g \rangle;$$

and if $L = L^{\sigma s}(\sqrt{d})$,

$$q_{x,y} = \langle 2\rangle\langle\langle d\rangle\rangle \sum_{[\sigma s, g]=s^{-1}t} \langle \omega_g\rangle.$$

Then $q_{x,y}$ is independent of the choice of standard generators, and we have in $\widetilde{W}(A, \sigma)$:

$$\langle x\rangle_\sigma \cdot \langle y\rangle_\sigma = (\mathrm{Tr}_{L^{\sigma s}/K})_*(q_{x,y}).$$

Proof. We have an isomorphism $f : (A, \sigma') \to (A, \sigma)$ in $\mathbf{Br}_h(K)$, given by $\langle x\rangle_\sigma$, where $\sigma'(a) = x^{-1}\sigma(a)x$ for all $a \in A$. We see that (A, σ') is naturally a $(G, L, \sigma s)$-crossed product. We have $f_*(\langle 1\rangle_{\sigma'}) = \langle x\rangle_\sigma$, and $f_*(\langle a\rangle_{\sigma'}) = \langle y\rangle_\sigma$ with $a = \varepsilon x^{-1}y$. Then $\langle x\rangle_\sigma \cdot \langle y\rangle_\sigma = T_{\sigma',a}$, and $a \in \mathcal{T}_{s^{-1}t}(A)$, so we can apply the previous proposition. The statement then follows, since $\omega_g = \sigma'(u_g)au_g$. □

Example 3.15. In particular, $\langle x\rangle_\sigma \cdot \langle y\rangle_\sigma = 0$ in $\widetilde{W}(A,\sigma)$ if $t = s\sigma$.

Remark 3.16. This shows that $\langle x\rangle_\sigma \cdot \langle y\rangle_\sigma$ is in the trace ideal of $\widetilde{W}(A,\sigma)$ relative to the étale algebra $L^{\sigma s}/K$, but also by symmetry in the trace ideal relative to $L^{\sigma t}/K$ (see remark 2.26).

3.5 Existence of symmetric standard generators

We still assume that (A,σ) is a (G,L,σ)-crossed product. We have seen that for any $t \in G$, a necessary condition for the existence of a non-zero symmetric element in $\mathcal{T}_t(A)$ is that $\bar{t} = t$. We want to establish exactly when such an element exists under this condition.

Proposition 3.17. *Let (A,σ) be a (G,L,σ)-crossed product, and let $t \in G$ be such that $\bar{t} = t$. If $t \neq \sigma$, then there is a symmetric invertible element in $\mathcal{T}_t(A)$.*

Proof. The statement means that if $\bar{t} = t$, we can choose some standard generator u_t such that $\mu_t = 1$. Note that in this situation, the element $\sigma_t \in \mathrm{Aut}_K(L)$ has order 2, the relation (19) becomes

$$\mu_t \sigma_t(\mu_t) = 1,$$

and taking $u_t' = c_t u_t$ gives

$$\mu_t' = \frac{\sigma_t(c_t)}{c_t}\mu_t.$$

We can then apply the Hilbert 90 theorem to the extension L/L^{σ_t} to see that there is $c \in L^*$ such that $\mu_t = \frac{c}{\sigma_t(c)}$, so indeed we can take $\mu_t' = 1$. □

We see that the situation is a little different when $t = \sigma$: then $\sigma_t = \mathrm{Id}_L$, so $\mu_t^2 = 1$ and μ_t does not depend on the choice of u_t.

Proposition 3.18. *Let (A,σ) be a (G,L,σ)-crossed product, such that $\sigma \in G$, and let us make a choice of standard generators on A. Then A is naturally a L-module on the left, and there is a canonical identification $A_L \simeq \mathrm{End}_L(A)$ given by $(a\otimes\lambda)\cdot x = ax\lambda$. We define*

$$\begin{array}{rccc} \varphi: & A\times A & \longrightarrow & L \\ & (x,y) & \longmapsto & \pi_\sigma(\sigma(x)y) \end{array}$$

where $\pi_\sigma(\sum_g u_g\lambda_g) = \lambda_\sigma$. Then φ is a L-bilinear form on A, which is μ_σ-symmetric, and its adjoint involution on A_L is σ_L.

Proof. By definition, for any $x \in A$ and any $a, b \in L$, we have

$$\pi_\sigma(axb) = \sigma(a)\pi_\sigma(x)b$$

and

$$\pi_\sigma(\sigma(x)) = \sigma(\mu_\sigma)\pi_\sigma(x).$$

so if $y \in A$:

$$\begin{aligned} \varphi(xa, yb) &= \pi_\sigma(\sigma(a)\sigma(x)yb) \\ &= a\varphi(x,y)b \end{aligned}$$

which shows that φ is L-bilinear. Furthermore:

$$\begin{aligned}\varphi(y,x) &= \pi_\sigma(\sigma(\sigma(x)y)) \\ &= \sigma(\mu_\sigma)\varphi(x,y)\end{aligned}$$

so φ is ε-symmetric with $\varepsilon = \sigma(\mu_\sigma)$ (we have $\varepsilon^2 = 1$ but that does not necessarily imply $\varepsilon = \pm 1$ if L is not a field). Finally, if $a, x, y \in A$ and $\lambda \in L$:

$$\begin{aligned}\varphi((a\otimes\lambda)\cdot x, y) &= \pi_\sigma(\sigma(\lambda)\sigma(x)\sigma(a)\cdot y) \\ &= \pi\sigma(\sigma(x)\cdot\sigma(a)y)\lambda \\ &= \varphi(x, \sigma_L(a\otimes\lambda)\cdot y)\end{aligned}$$

so $\sigma_\varphi = \sigma_L$. This shows in particular that $\varepsilon = \pm 1$, so $\varepsilon = \sigma(\mu_\sigma) = \mu_\sigma$. □

Corollary 3.19. *Let (A,σ) be a (G,L,σ)-crossed product, such that $\sigma \in G$. Then $\mathcal{T}_\sigma(A) \subset \mathrm{Sym}(A,\sigma)$ if σ is orthogonal, and $\mathcal{T}_\sigma(A) \subset \mathrm{Skew}(A,\sigma)$ if σ is symplectic. In particular, if σ acts trivially on L then σ is orthogonal.*

Proof. If σ is orthogonal, then so is σ_L, so the bilinear form φ in proposition 3.18 must be symmetric. This means that any invertible element $u_\sigma \in \mathcal{T}_\sigma(A)$ is symmetric (since $\mu_\sigma = 1$). But then any $a \in \mathcal{T}_\sigma(A)$ has the form λu_σ for some $\lambda \in L$, so $\sigma(a) = \sigma(u_\sigma)\sigma(\lambda) = a$. The reasoning is the same when σ is symplectic. □

Remark 3.20. In [11, 4.13], it is shown that if an involution on A acts trivially on any subfield L of A with $[L:K] = \deg(A)$, then σ must be orthogonal. The proof can be adapted to the case where L is an étale subalgebra (not necessarily Galois).

3.6 Mixed Witt ring of quaternions

As a special case, we can give a complete description of the product in the mixed Witt ring of a quaternion algebra with its canonical involution (of course we could have done a direct computation in this case, which is much simpler than the general case).

Lemma 3.21. *Let $L = K(\sqrt{d})$ be an étale quadratic algebra, and let $a \in L^*$. If $a \in K\sqrt{d}$ then $(\mathrm{Tr}_{L/K})_*(\langle a\rangle)$ is hyperbolic, and otherwise*

$$(\mathrm{Tr}_{L/K})_*(\langle a\rangle) = \langle \mathrm{Tr}_{L/K}(a)\rangle\langle\langle -d\cdot N_{L/K}(a)\rangle\rangle.$$

Proof. If $a = a_0 + a_1\sqrt{d}$, then in the basis $(1,\sqrt{d})$ the matrix of $(\mathrm{Tr}_{L/K})_*(\langle a\rangle)$ is

$$\begin{pmatrix} 2a_0 & 2da_1 \\ 2da_1 & 2da_0 \end{pmatrix},$$

so if $a_0 = 0$ it is hyperbolic, and otherwise it is isometric to $\langle 2a_0\rangle\langle\langle -\Delta\rangle\rangle$ where Δ is the determinant of the matrix. Now $\Delta = 4d(a_0^2 - da_1^2)$, so we can conclude since $\mathrm{Tr}_{L/K}(a) = 2a_0$ and $N_{L/K}(a) = a_0^2 - da_1^2$. □

Recall that if Q is a quaternion algebra, then its reduced norm map is a quadratic form on Q, denoted $n_Q \in GW(K)$, and it is the unique 2-Pfister form whose Clifford invariant $e_2(n_Q) \in H^2(K,\mu_2)$ is the Brauer class of Q.

For any pure quaternions $z_1, z_2 \in Q$, the Brauer class $[Q]$ and the symbol $(z_1^2, z_2^2) \in H^2(K, \mu_2)$ have a common slot (for instance z_1^2), so $[Q] + (z_1^2, z_2^2)$ is a symbol. We write $\varphi_{z_1,z_2} \in GW(K)$ for the unique 2-Pfister form whose Clifford invariant is this symbol. In particular, if z_1 and z_2 anti-commute, φ_{z_1,z_2} is hyperbolic.

Proposition 3.22. *Let (Q, γ) be a quaternion algebra over K endowed with its canonical symplectic involution. Then for any $a, b \in K^*$ we have*

$$\langle a\rangle_\gamma \cdot \langle b\rangle_\gamma = \langle 2ab\rangle n_Q \in GW(K)$$

in $\widetilde{GW}(Q, \gamma)$. Furthermore, for any invertible pure quaternions $z_1, z_2 \in Q$,

$$\langle z_1\rangle_\gamma \cdot \langle z_2\rangle_\gamma = \langle -\operatorname{Trd}_Q(z_1z_2)\rangle\varphi_{z_1,z_2} \in GW(K)$$

in $\widetilde{GW}(Q, \gamma)$ (if z_1 and z_2 anti-commute, then $\operatorname{Trd}_Q(z_1z_2) = 0$ so $\langle -\operatorname{Trd}_Q(z_1z_2)\rangle$ is not well-defined, and in that case we mean that $\langle z_1\rangle_\gamma \cdot \langle z_2\rangle_\gamma$ is hyperbolic).

Proof. Note that for any choice of invertible pure quaternion z in Q, $L = K(z)$ is a G-Galois algebra in Q of maximal dimension where $G = \{1, \sigma\} \simeq \mathbb{Z}/2\mathbb{Z}$, and (Q, γ) is a (G, L, σ)-crossed product.

Of course if $a, b \in K^*$, then $a, b \in \mathcal{T}_1(A)$, so we can use corollary 3.14 with $s = t = 1$, and take for u_σ any invertible pure z' that anti-commutes with z. Then we find $\langle a\rangle_\gamma \cdot \langle b\rangle_\gamma = \langle 2\rangle\langle\langle z^2\rangle\rangle\langle\omega_1, \omega_\sigma\rangle$ where $\omega_1 = a^{-1}b$ and $\omega_\sigma = -a^{-1}b(z')^2$, so we can conclude since $\langle\langle z^2, (z')^2\rangle\rangle = n_Q$.

Now given z_1 and z_2, take for z any pure quaternion that anti-commutes with both z_1 and z_2. Then $z_1, z_2 \in \mathcal{T}_\sigma(A)$, so we can apply corollary 3.14 with $s = t = \sigma$, so $\langle z_1\rangle_\gamma \cdot \langle z_2\rangle_\gamma = \operatorname{Tr}_{L/K}(q_{z_1,z_2})$. To compute $q_{z_1,z_2} = \langle\omega_1, \omega_\sigma\rangle$, we may take $u_\sigma = z_1$, so $\omega_1 = -z_1^{-1}z_2$ and $\omega_\sigma = z_2z_1$. Thus $q_{z_1,z_2} = \langle\langle z_1^2\rangle\rangle\langle -z_1z_2\rangle$ (with indeed $z_1z_2 \in L^*$). According to lemma 3.21, $\operatorname{Tr}_{L/K}(q_{z_1,z_2})$ is hyperbolic if z_1 and z_2 anti-commute (which is equivalent to $z_1z_2 \in Kz$), and otherwise it is equal to $\langle\langle z_1^2\rangle\rangle \cdot \langle\operatorname{Tr}_{L/K}(-z_1z_2)\rangle\langle\langle -z^2 \cdot N_{L/K}(-z_1z_2)\rangle\rangle$. Then we can conclude since $\operatorname{Tr}_{L/K}(z_1z_2) = \operatorname{Trd}_Q(z_1z_2)$, $N_{L/K}(-z_1z_2) = z_1^2z_2^2$, and

$$e_2(\langle\langle z_1^2, -z^2z_1^2z_2^2\rangle\rangle) = (z_1^2, z_2^2) + (z_1^2, z^2) = (z_1^2, z_2^2) + [Q]. \qquad \square$$

Remark 3.23. The twisted involution trace forms of (Q, γ) are computed in a much more straightforward way in [11, 11.6]. We wanted to showcase how to use the more general corollary 3.14 in a concrete case. The result can also be obtained using proposition 5.10.

4 Lambda-operations

In the classical theory of quadratic forms, the Witt ring $W(K)$ is usually more practical than the more fundamental Grothendieck-Witt ring $GW(K)$, and the loss on information between the two is small enough that $W(K)$ sees much more use. But there is at least one important feature that $GW(K)$ enjoys and not $W(K)$: it is a λ-ring (see [24] for a reference on λ-rings, and [16] for a proof that $GW(K)$ is a λ-ring). Indeed, given a bilinear form (V, b), we may define its λ-powers $(\Lambda^d(V), \lambda^d(b))$, setting

$$\lambda^d(b)(u_1 \wedge \cdots \wedge u_d, v_1 \wedge \cdots \wedge v_d) = det(b(u_i, v_j)).$$

We want to extend this structure to $\widetilde{GW}(A,\sigma)$, which is related to the construction of λ-powers of an algebra with involution given in [11, 10.A].

4.1 Alternating powers of a module

If V is K-vector space, since $char(K) \neq 2$ we may see the exterior power $\Lambda^d(V)$ in two different ways: either as a quotient of $V^{\otimes d}$ (which is the canonical construction), or as a subspace. Precisely, for any $\pi \in \mathfrak{S}_d$ we set

$$g_\pi : \begin{array}{ccc} V^{\otimes d} & \longrightarrow & V^{\otimes d} \\ v_1 \otimes \cdots \otimes v_d & \longmapsto & v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(d)}, \end{array}$$

and we define the anti-symmetrization map

$$s_d = \sum_{\pi \in \mathfrak{S}_d} (-1)^\pi g_\pi$$

where $(-1)^\pi$ is the signature of the permutation π. Then the map s_d is alternating, so by universal property we get an induced map $\Lambda^d(V) \to \mathrm{Alt}^d(V)$, where $\mathrm{Alt}^d(V) \subset V^{\otimes d}$ is the image of s_d, and a classical result of linear algebra states that this is an isomorphism, that may be explicited as $v_1 \wedge \cdots \wedge v_d \mapsto s_d(v_1 \otimes \cdots \otimes v_d)$.

Since the maps g_π are precisely the elements $g_A(\pi)$ in the case $A = \mathrm{End}_K(V)$ (see remark 1.2), we may generalize this construction in the non-split case. Thus we define, as in [11, §10.A], the anti-symmetrisation element $s_{d,A} \in A^{\otimes d}$ by:

$$s_{d,A} = \sum_{\pi \in \mathfrak{S}_d} (-1)^\pi g_A(\pi). \tag{22}$$

Note that $s_{0,A} = 1 \in K$, $s_{1,A} = 1 \in A$, and $s_{2,A} = 1 - g_A \in A \otimes_K A$.

Definition 4.1. *Let A be a central simple algebra over K, let V be a right A-module, with $B = \mathrm{End}_A(V)$, and let $d \in \mathbb{N}$. We set*

$$\mathrm{Alt}^d(V) = s_{d,B} V^{\otimes d} \subset V^{\otimes d}$$

as a right $A^{\otimes d}$-module, with in particular $\mathrm{Alt}^0(V) = K$ and $\mathrm{Alt}^1(V) = V$.

By construction, we retrieve the case of vector spaces discussed above, when $A = K$. In general, we have:

Proposition 4.2. *Let A be a central simple algebra over K, let V be a right A-module, and let $d \in \mathbb{N}$. Then*

$$\mathrm{rdim}_{A^{\otimes d}}(\mathrm{Alt}^d(V)) = \binom{\mathrm{rdim}_A(V)}{d}.$$

Proof. It is enough to check this when A is split, in which case $A = \mathrm{End}_K(U)$ for some K-vector space U, and $V \simeq W \otimes_K U$ for some K-vector space W, so that $\mathrm{End}_A(V) \simeq \mathrm{End}_K(W)$. Then by construction $\mathrm{Alt}^d(V) \simeq \Lambda^d(W) \otimes_K U^{\otimes d}$, so $\mathrm{End}_{A^{\otimes d}}(V) \simeq \mathrm{End}_K(\lambda^d(W))$. Thus if $n = \dim_K(W)$, then $\mathrm{rdim}_A(V) = n$ and $\mathrm{rdim}_{A^{\otimes d}}(\mathrm{Alt}^d(V)) = \binom{n}{d}$. ☐

In particular, if $d > \mathrm{rdim}_A(V)$ then $\mathrm{Alt}^d(V) = \{0\}$. This means that we have to be a little careful if we want to allow arbitrary $d \in \mathbb{N}$, since most of our results have been stated for non-zero modules. Note that this means that if $d > \deg(A)$ then $s_{d,A} = 0$.

4.2 The shuffle product

We start by recalling some elementary results about symmetric groups and shuffles. Let $d \in \mathbb{N}$, and $p, q \in \mathbb{N}$ such that $p + q = d$. Then we can define the Young subgroup $\mathfrak{S}_{p,q} \subset \mathfrak{S}_d$, which contains the permutations that preserve the sets $\{1, \dots, p\}$ and $\{p+1, \dots, p+q\}$. There is a natural isomorphism $\mathfrak{S}_{p,q} \simeq \mathfrak{S}_p \times \mathfrak{S}_q$, such that the restriction of the signature of $\mathfrak{S}_d$ corresponds to the product of the signatures on $\mathfrak{S}_p$ and $\mathfrak{S}_q$.

Lemma 4.3. *Let V_1 and V_2 be right A-modules, and let $B_i = \mathrm{End}_A(V_i)$ and $B = \mathrm{End}_A(V_1 \oplus V_2)$. Take $\pi \in \mathfrak{S}_{p,q}$, corresponding to the product of $\pi_1 \in \mathfrak{S}_p$ and $\pi_2 \in \mathfrak{S}_q$. Let $x \in V_1^{\otimes p}$ and $y \in V_2^{\otimes q}$. Then:*

$$g_B(\pi) \cdot (x \otimes y) = (g_{B_1}(\pi_1) \cdot x) \otimes (g_{B_2}(\pi_2) \cdot y).$$

Proof. It is enough to treat the case where A is split, in which case it is clear by construction. □

We may also define the set of (p,q)-shuffles $Sh(p,q) \subset \mathfrak{S}_d$, which are the permutations that are increasing functions when restricted to $\{1, \dots, p\}$ and $\{p+1, \dots, p+q\}$.

Lemma 4.4. *Any element of $\mathfrak{S}_d$ can be written in a unique way as $\pi\sigma$, with $\pi \in Sh(p,q)$ and $\sigma \in \mathfrak{S}_{p,q}$.*

Proof. Let $\tau \in \mathfrak{S}_d$. We set $\sigma_1 \in \mathfrak{S}_p$ and $\sigma_2 \in \mathfrak{S}_q$ defined by $\tau(\sigma_1^{-1}(1)) < \cdots < \tau(\sigma_1^{-1}(p))$ and $\tau(\sigma_2^{-1}(p+1)) < \cdots < \tau(\sigma_2^{-1}(p+q))$; in other words, σ_1 is obtained by ordering $\tau(1), \dots, \tau(p)$ in increasing order, and likewise for σ_2 with $\tau(p+1), \dots, \tau(p+q)$. We take $\sigma \in \mathfrak{S}_{p,q}$ corresponding to (σ_1, σ_2), and $\pi = \tau\sigma^{-1}$. Then by construction $\pi(1) < \cdots < \pi(p)$ and $\pi(p+1) < \cdots < \pi(p+q)$, so $\pi \in Sh(p,q)$.

If we have another decomposition $\tau = \sigma'\pi'$, then since $\pi' \in Sh(p,q)$ we must have $\tau((\sigma_1')^{-1}(1)) < \cdots < \tau((\sigma_1')^{-1}(p))$ and $\tau((\sigma_2')^{-1}(p+1)) < \cdots < \tau((\sigma')_2^{-1}(p+q))$, so $\sigma' = \sigma$ (and $\pi' = \pi$). □

As a consequence, we get:

Lemma 4.5. *Let A be a central simple algebra over K. Then the element*

$$sh_{p,q,A} = \sum_{\pi \in Sh(p,q)} (-1)^{\pi} g_A(\pi) \in A^{\otimes d}$$

satisfies

$$sh_{p,q,A} \cdot (s_{p,A} \otimes s_{q,A}) = s_{d,A}.$$

Proof. We have, using lemmas 4.3 and 4.4:

$$\begin{aligned} sh_{p,q}(s_p \otimes s_q) &= \sum_{\pi \in Sh(p,q)} (-1)^{\pi} g(\pi)(s_p \otimes s_q) \\ &= \sum_{\pi \in Sh(p,q)} (-1)^{\pi} g(\pi) \left(\sum_{\sigma \in \mathfrak{S}_{p,q}} (-1)^{\sigma} g(\sigma) \right) \\ &= \sum_{\pi \in Sh(p,q), \sigma \in \mathfrak{S}_{p,q}} (-1)^{\pi\sigma} g(\pi\sigma) \\ &= s_d. \end{aligned}$$

□

Let A be a central simple algebra over K, and V a A-module, with $B = \mathrm{End}_V(A)$. If $p + q = d$, we define a $A^{\otimes d}$-module morphism

$$V^{\otimes p} \otimes_K V^{\otimes q} \to V^{\otimes d},$$

called the *shuffle product* and denoted $x\#y$, by

$$x\#y = sh_{p,q,B} \cdot (x \otimes y) \tag{23}$$

where $sh_{p,q,B}$ is defined in lemma 4.5.

We easily see be definition that the shuffle product is associative and alternating, and in particular anti-symmetric.

Proposition 4.6. *The shuffle product induces a commutative diagram*

$$\begin{array}{ccc} V^{\otimes p} \otimes_K V^{\otimes q} & \xrightarrow{\otimes} & V^{\otimes d} \\ \downarrow & & \downarrow \\ \mathrm{Alt}^p(V) \otimes_K \mathrm{Alt}^q(V) & \xrightarrow{\#} & \mathrm{Alt}^d(V). \end{array}$$

Proof. Unwrapping the definitions, this is precisely equivalent to lemma 4.5. □

We now establish the analogue of the well-known addition formula for exterior powers of vector spaces:

Proposition 4.7. *Let U and V be right A-modules. Then for any $d \in \mathbb{N}$ the shuffle product induces an isomorphism of $A^{\otimes d}$-modules :*

$$\bigoplus_{k=0}^{d} \mathrm{Alt}^k(U) \otimes_K \mathrm{Alt}^{d-k}(V) \xrightarrow{\sim} \mathrm{Alt}^d(U \oplus V).$$

Proof. Using the previous proposition, we easily establish that $\mathrm{Alt}^d(U \oplus V)$ is linearly spanned by the elements of the type $x_1\#\cdots\#x_d$ with x_i in U or V. Now since the shuffle product is anti-symmetrical, we can permute the x_i so that $x_1, \dots, x_k \in U$ and $x_{k+1}, \dots, x_d \in V$. But any element of this type is obviously in the image of the map described in the statement of the proposition, so this map is surjective. We may then conclude that it is an isomorphism by checking the dimensions over K. □

Note that by construction, Alt^d is a covariant functor with respect to bimodule isomorphisms: if V, W are B-A-bimodules, and $f : V \to W$ is an isomorphism, there is a unique isomorphism $\mathrm{Alt}^d(f)$ of $B^{\otimes d}$-$A^{\otimes d}$-bimodules that makes this diagram commute:

$$\begin{array}{ccc} \mathrm{Alt}^d(V) & \xrightarrow{\mathrm{Alt}^d(f)} & \mathrm{Alt}^d(W) \\ {\scriptstyle j_V}\downarrow & & \downarrow{\scriptstyle j_W} \\ V^{\otimes d} & \xrightarrow{f} & W \end{array}$$

where j_V, j_W are the canonical inclusions.

4.3 Alternating powers of a ε-hermitian form

Now if V is a A-module equipped with a ε-hermitian form h with respect to some involution σ on A, we want to endow $\mathrm{Alt}^d(V)$ with an induced form $\mathrm{Alt}^d(h)$ such that in the split case we recover the exterior power of the bilinear form.

Lemma 4.8. *Let A be a central simple algebra over K. Then the element $s_{d,A} \in A^{\otimes d}$ is symmetric for the involution $\sigma^{\otimes d}$ for any involution σ on A.*

Proof. It follows directly from the fact, proved in lemma 1.3, that the Goldman element is symmetric for $\sigma \otimes \sigma$. □

This observation allows the following definition:

Definition 4.9. *Let (A, σ) be an algebra with involution over K, and let (V, h) be a ε-hermitian module over (A, σ), with $B = \mathrm{End}_A(V)$. We set:*

$$\begin{array}{rrcl} \mathrm{Alt}^d(h): & \mathrm{Alt}^d(V) \times \mathrm{Alt}^d(V) & \longrightarrow & A^{\otimes d} \\ & (s_{d,B}x, s_{d,B}y) & \longmapsto & h^{\otimes d}(x, s_{d,B}y) = h^{\otimes d}(s_{d,B}x, y). \end{array}$$

The equality on the right is a consequence of the fact that $s_{d,B}$ is symmetric for $\tau^{\otimes d}$ where τ is the adjoint involution of h. The left-hand side of the equality shows that the map is well-defined in $s_{d,B}y$, and the right-hand side shows that it is well-defined in $s_{d,B}x$.

Proposition 4.10. *The application $\mathrm{Alt}^d(h)$ is a ε^d-hermitian form over $(A^{\otimes d}, \sigma^{\otimes d})$.*

Proof. We have for all $x, y \in V^{\otimes d}$ and all $a, b \in A^{\otimes d}$:

$$\begin{aligned} \mathrm{Alt}^d(h)(s_{d,B}x \cdot a, s_{d,B}y \cdot b) &= h^{\otimes d}(xa, s_{d,B}yb) \\ &= \sigma^{\otimes d}(a) h^{\otimes d}(x, s_{d,B}y) b \\ &= \sigma^{\otimes d}(a) \, \mathrm{Alt}^d(h)(s_{d,B}x, s_{d,B}y) b \end{aligned}$$

and

$$\begin{aligned} \mathrm{Alt}^d(h)(s_{d,B}y, s_{d,B}x) &= h^{\otimes d}(y, s_{d,B}x) \\ &= \varepsilon^d \sigma^{\otimes d}(h^{\otimes d}(s_{d,B}x, y)) \\ &= \varepsilon^d \sigma^{\otimes d}(\mathrm{Alt}^d(h)(s_{d,B}x, s_{d,B}y)). \end{aligned}$$ □

Remark 4.11. When $d > \mathrm{rdim}_A(V)$, $\mathrm{Alt}^d(h)$ is the trivial hermitian form on the zero module, so $(\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$ defines a morphism in $\mathbf{Br}_h(K)'$ and not $\mathbf{Br}_h(K)$.

Remark 4.12. We could have exhibited a canonical isomorphism $\mathrm{Alt}^d(V^*) \simeq \mathrm{Alt}^d(V)^*$, and then defined $\mathrm{Alt}^d(h)$ by imposing that $\widehat{\mathrm{Alt}^d(h)}$ be the composition of $\mathrm{Alt}^d(\hat{h})$ with this canonical identification.

Remark 4.13. Since we defined $\mathrm{Alt}^d(V)$ as a submodule of $V^{\otimes d}$, in addition to $\mathrm{Alt}^d(h)$ it is also naturally equipped with the restriction of $h^{\otimes d}$, and we may wonder what the link is between the two. Since $h^{\otimes d}(s_{d,B}x, s_{d,B}y) = h^{\otimes d}(x, s_{d,B}^2 y)$ and $s_{d,B}^2 = d! s_{d,B}$ (which is easy to see from the definition), we can conclude that

$$h^{\otimes d}_{|\,\mathrm{Alt}^d(V)} = \langle d! \rangle \, \mathrm{Alt}^d(h).$$

In particular, in arbitrary characteristic we cannot simply define $\mathrm{Alt}^d(h)$ in terms of the restriction of $h^{\otimes d}$.

We can then show the compatibility of this construction with the sum formula:

Proposition 4.14. *Let (A,σ) be an algebra with involution over K, and let (U,h) and (V,h') be ε-hermitian modules over (A,σ). The module isomorphism in proposition 4.7 induces an isometry*

$$\bigoplus_{k=0}^{d} \mathrm{Alt}^k(h) \otimes_K \mathrm{Alt}^{d-k}(h') \xrightarrow{\sim} \mathrm{Alt}^d(h \perp h').$$

Proof. We set $B_1 = \mathrm{End}_A(U)$, $B_2 = \mathrm{End}_A(V)$, and $B = \mathrm{End}_A(U \oplus V)$. Let $u, u' \in U^{\otimes k}$ and $v, v' \in V^{\otimes d-k}$. Then

$$\begin{aligned}
&\mathrm{Alt}^d(h \perp h')((s_{k,B_1}u)\#(s_{d-k,B_2}v), (s_{k,B_1}u)\#(s_{d-k,B_2}v)) \\
=&\mathrm{Alt}^d(h \perp h')(s_{d,B}(u \otimes v), s_{d,B}(u' \otimes v')) \\
=&(h \perp h')^{\otimes d}(s_{d,B}(u \otimes v), u' \otimes v') \\
=&\sum_{\pi \in \mathfrak{S}_d} (-1)^{\pi} (h \perp h')^{\otimes d}(g_B(\pi)(u \otimes v), u' \otimes v'),
\end{aligned}$$

where we used lemma 4.4 for the first equality. We want to show that

$$(h \perp h')^{\otimes d}(g_B(\pi)(u \otimes v), u' \otimes v')$$

is zero if $\pi \notin \mathfrak{S}_{k,d-k}$. But if $u = x_1 \otimes \cdots \otimes x_k$, $u' = y_1 \otimes \cdots \otimes y_k$, and $v = x_{k+1} \otimes \cdots \otimes x_d$, $v' = y_{k+1} \otimes \cdots \otimes y_d$, then using lemma 1.1:

$$\begin{aligned}
&(h \perp h')^{\otimes d}(g_B(\pi)(u \otimes v), u' \otimes v') \\
&= (h \perp h')^{\otimes d}((x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(d)}) \cdot g_A(\pi), (y_1 \otimes \cdots \otimes y_d)) \\
&= \sigma^{\otimes d}(g_A(\pi)) \cdot (h \perp h')(x_{\pi^{-1}(1)}, y_1) \otimes \cdots \otimes (h \perp h')(x_{\pi^{-1}(d)}, y_d)
\end{aligned}$$

which is indeed zero if $\pi \notin \mathfrak{S}_{k,d-k}$ since at least one of the $(h \perp h')(x_{\pi^{-1}(i)}, y_i)$ will be zero. Hence:

$$\begin{aligned}
&\mathrm{Alt}^d(h \perp h')(s_{k,B_1}u\#s_{d-k,B_2}v, s_{k,B_1}u\#s_{d-k,B_2}v) \\
&= \sum_{\pi \in \mathfrak{S}_{k,d-k}} (-1)^{\pi} (h \perp h')^{\otimes d}(g_B(\pi)(u \otimes v), u' \otimes v') \\
&= \sum_{\pi_1 \in \mathfrak{S}_k} \sum_{\pi_2 \in \mathfrak{S}_{d-k}} (-1)^{\pi_1\pi_2} (h \perp h')^{\otimes d}(g_{B_1}(\pi_1)u \otimes g_{B_2}(\pi_2)v), u' \otimes v') \\
&= h(s_{k,B_1}u, u') \otimes h'(s_{d-k,B_2}v, v').
\end{aligned}$$

□

Starting from some morphism $f : (B,\tau) \to (A,\sigma)$ in $\mathbf{Br}_h(K)$, we have defined another morphism $\mathrm{Alt}^d(f)$, in $\mathbf{Br}_h(K)'$ when d is large enough, with target $(A^{\otimes d}, \sigma^{\otimes d})$. It is natural to try to understand what is the source object of this morphism; in other words, we want to study $\mathrm{End}_{A^{\otimes d}}(\mathrm{Alt}^d(V))$ and $\sigma_{\mathrm{Alt}^d(h)}$ in terms of (V,h). First we take a look at the special case of identity morphisms in $\mathbf{Br}_h(K)$.

Definition 4.15. *Let (A,σ) be an algebra with inveolution over K, and let $d \in \mathbb{N}$. We write $\Lambda^d_\sigma = \mathrm{Alt}^d(\langle 1 \rangle_\sigma)$, where we recall that $\langle 1 \rangle_\sigma$ is the identity of (A,σ) in $\mathbf{Br}_h(K)$.*

Then $(\lambda^d(A), \sigma^{\wedge d})$ *is the algebra with involution such that*

$$\Lambda^d_\sigma : (\lambda^d(A), \sigma^{\wedge d}) \to (A^{\otimes d}, \sigma^{\otimes d})$$

is a morphism in $\mathbf{Br}_h(K)$.

Note that this definition agrees with the one given in [11], ours being a reformulation in the language of $\mathbf{Br}_h(K)$ (the main difference is that in [11] $\sigma^{\wedge d}$ is defined directly and not as the adjoint involution of some hermitian form). The algebra $\lambda^d(A)$ is actually well-defined with no reference to any involution, simply by $\lambda^d(A) = \mathrm{End}_{A^{\otimes d}}(\mathrm{Alt}^d(A))$ where A is seen as a tautological A-module. If $d > \deg(A)$, then $\Lambda^d(A)$ is the zero ring.

Proposition 4.16. *Let* $f : (B, \tau) \to (A, \sigma)$ *be a morphism in* $\mathbf{Br}_h(K)$*. Then for any* $d \in \mathbb{N}$ *the following diagram in* $\mathbf{Br}_h(K)'$ *commutes:*

$$\begin{array}{ccc} (\lambda^d(B), \tau^{\wedge d}) & & \\ \Big\downarrow \Lambda^d_\tau & \searrow^{\mathrm{Alt}^d(f)} & \\ (B^{\otimes d}, \tau^{\otimes d}) & \xrightarrow{f^{\otimes d}} & (A^{\otimes d}, \sigma^{\otimes d}). \end{array}$$

Proof. Say f corresponds to the ε-hermitian module (V, h). By definition, $f^{\otimes d} \circ \Lambda^d_\tau$ then corresponds to $(s_{d,B} B^{\otimes d} \otimes_{B^{\otimes d}} V^{\otimes d}, h^{\otimes d} \circ \mathrm{Alt}^d(\langle 1 \rangle_\tau))$, where $s_{d,B} B^{\otimes d} \otimes_{B^{\otimes d}} V^{\otimes d} \simeq s_{d,B} V^{\otimes d} = \mathrm{Alt}^d(V)$ and $h^{\otimes d} \circ \mathrm{Alt}^d(\langle 1 \rangle_\tau)$ is

$$((s_{d,B} x) \otimes u, (s_{d,B} y) \otimes v) \mapsto h^{\otimes d}(u, \sigma^{\otimes d}(x) s_{d,B} y v)$$

which under the above identification (taking $x = y = 1$) is exactly $\mathrm{Alt}^d(h)$. □

Corollary 4.17. *Let* f *and* g *be two morphisms in* $\mathbf{Br}_h(K)$ *such that* $f \circ g$ *exists. Then for any* $d \in \mathbb{N}$*, we have in* $\mathbf{Br}_h(K)'$*:*

$$f^{\otimes d} \circ \mathrm{Alt}^d(g) = \mathrm{Alt}^d(f \circ g).$$

Proof. Write $f : (B, \tau) \to (A, \sigma)$ and $g : (C, \theta) \to (B, \tau)$. The result follows from the fact that the following diagram in $\mathbf{Br}_h(K)'$ commutes, which is established by two applications of proposition 4.16:

$$\begin{array}{ccccc} (\lambda^d(C), \theta^{\wedge d}) & & & & \\ \Lambda^d_\theta \Big\downarrow & \searrow^{\mathrm{Alt}^d(g)} & & \searrow^{\mathrm{Alt}^d(f \circ g)} & \\ (C^{\otimes d}, \theta^{\otimes d}) & \xrightarrow[g^{\otimes d}]{} & (B^{\otimes d}, \tau^{\otimes d}) & \xrightarrow[f^{\otimes d}]{} & (A^{\otimes d}, \sigma^{\otimes d}). \end{array}$$ □

We will not use it in the rest of the article, but it is interesting to note that there is a functorial behaviour to the construction of $(\lambda^d(A), \sigma^{\wedge d})$.

Definition 4.18. *Let* $f : (B, \tau) \to (A, \sigma)$ *be a morphism in* $\mathbf{Br}_h(K)$*. Then for any* $d \in \mathbb{N}$ *we define* $f^{\wedge d}$ *as the unique morphism in* $\mathbf{Br}_h(K)'$ *such that the following square commutes:*

$$\begin{array}{ccc} (\lambda^d(B), \tau^{\wedge d}) & \xrightarrow{f^{\wedge d}} & (\lambda^d(A), \sigma^{\wedge d}) \\ \Big\downarrow \Lambda^d_\tau & & \Big\downarrow \Lambda^d_\sigma \\ (B^{\otimes d}, \tau^{\otimes d}) & \xrightarrow{f^{\otimes d}} & (A^{\otimes d}, \sigma^{\otimes d}). \end{array}$$

If $f = (V, h)$ *we write* $f^{\wedge d} = (V^{\wedge d}, h^{\wedge d})$.

Remark 4.19. Explicitly, we have $V^{\wedge d} \simeq s_{d,B} V^{\otimes d} s_{d,A}$, and the endomorphism $h^{\wedge d}(s_{d,B} x s_{d,A}, s_{d,B} y s_{d,A}) \in \lambda^d(A)$ of $\mathrm{Alt}^d(A) = s_{d,A} A^{\otimes d}$ is the multiplication on the left by $s_{d,A} h^{\otimes d}(x, s_{d,B} y)$.

Proposition 4.20. *The association $(A,\sigma) \mapsto (\lambda^d(A), \sigma^{\wedge d})$ and $f \mapsto f^{\wedge d}$ defines a functor from $\mathbf{Br}_h(K)$ to $\mathbf{Br}_h(K)'$, that has a natural extension to an endofunctor of $\mathbf{Br}_h(K)'$.*

Proof. Any functor $\mathbf{Br}_h(K) \to \mathbf{Br}_h(K)'$ has an obvious extension to $\mathbf{Br}_h(K)'$ which preserves the zero object and the zero morphisms.

The functoriality is immediate from the fact that the following diagrams commute in $\mathbf{Br}_h(K)'$:

$$\begin{array}{ccc} (\lambda^d(A), \sigma^{\wedge d}) & \xrightarrow{\langle 1\rangle_{\sigma^{\wedge d}}} & (\lambda^d(A), \sigma^{\wedge d}) \\ \downarrow \Lambda^d_\sigma & & \downarrow \Lambda^d_\sigma \\ (A^{\otimes d}, \sigma^{\otimes d}) & \xrightarrow{\langle 1\rangle_{\sigma^{\otimes d}}} & (A^{\otimes d}, \sigma^{\otimes d}) \end{array}$$

and

$$\begin{array}{ccccc} (\lambda^d(C), \theta^{\wedge d}) & \xrightarrow{g^{\wedge d}} & (\lambda^d(B), \tau^{\wedge d}) & \xrightarrow{f^{\wedge d}} & (\lambda^d(A), \sigma^{\wedge d}) \\ \downarrow \Lambda^d_\theta & & \downarrow \Lambda^d_\tau & & \downarrow \Lambda^d_\sigma \\ (C^{\otimes d}, \theta^{\otimes d}) & \xrightarrow{g^{\otimes d}} & (B^{\otimes d}, \tau^{\otimes d}) & \xrightarrow{f^{\otimes d}} & (A^{\otimes d}, \sigma^{\otimes d}). \end{array} \quad \square$$

4.4 Exterior powers of a ε-hermitian module

In the case of vector spaces and bilinear forms, $(\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$ gave an appropriate definition for an operation $\lambda^d : SW(K) \to SW(K)'$, but for a general A it simply defines a map $SW^\varepsilon(A,\sigma) \to SW^{\varepsilon^d}(A^{\otimes d}, \sigma^{\otimes d})'$. We can then get back to (A,σ) using the isomorphism $\varphi^{(d)}_{(A,\sigma)}$ in $\mathbf{Br}_h(K)$ (recall definition 1.17). Precisely:

Definition 4.21. *Let (A,σ) be an algebra with involution over K, and let $(V,h) \in SW^\varepsilon(A,\sigma)$. We set*

$$(\Lambda^d(V), \lambda^d(h)) = \varphi^{(d)}_{(A,\sigma)} \circ (\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$$

in $\mathbf{Br}_h(K)'$. This defines a map $\lambda^d : SW^\varepsilon(A,\sigma) \to SW(K)'$ if d is even, and $\lambda^d : SW^\varepsilon(A,\sigma) \to SW^\varepsilon(A,\sigma)'$ if d is odd.

Remark 4.22. Note that $\Lambda^d(V)$ depends on σ, even though V and $\mathrm{Alt}^d(V)$ are defined with no reference to any involution. On the other hand, $\Lambda^d(V)$ does not depend on h. See proposition 4.33 for an illustration of this. Furthermore, if $d > \mathrm{rdim}_A(V)$, then $\Lambda^d(V) = 0$.

Remark 4.23. If $(A,\sigma) = (K, \mathrm{Id})$, then $(\Lambda^d(V), \lambda^d(h)) \simeq (\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$ as bilinear spaces, and they coincide with the classical definition of exterior powers, but we have to pay attention to the grading. If (V,h) is in the odd component, then we have to distinguish two cases. When d is odd, $(\Lambda^d(V), \lambda^d(h))$ and $(\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$ correspond to the same element in the odd component. On the other hand, when d is even, then $(\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$ is still in the odd component, while $(\Lambda^d(V), \lambda^d(h))$ corresponds to the copy of that element in the even component. See remark 4.29 for further comments.

We want to show that this gives the structure we wanted on $\widetilde{GW}(A,\sigma)$. Recall (see [24]) that a pre-λ-ring is a commutative ring R endowed with maps $\lambda^d : R \to R$ for all $d \in \mathbb{N}$ such that for all $x, y \in R$, $\lambda^0(x) = 1$, $\lambda^1(x) = x$, and

$$\lambda^d(x+y) = \sum_{k=0}^{d} \lambda^k(x)\lambda^{d-k}(y).$$

Any family $(\lambda^d)_d$ of functions $R \to R$ can be encoded as a function $\lambda_t : R \to R[[t]]$, with $\lambda_t(x) = \sum_{d\in\mathbb{N}} \lambda^d(x)t^d$.

If we define $\Lambda(R) = 1 + tR[[t]]$ as the set of formal power series with constant coefficient 1, then $\Lambda(R)$ is a multiplicative subgroup of $R[[t]]^*$, with a group morphism $\eta : (\Lambda(R), \cdot) \to (R, +)$ which sends a formal series to its degree 1 coefficient. Then $(\lambda^d)_{d\in\mathbb{N}}$ defines a pre-λ-ring structure iff λ_t is a group morphism with η as a section.

Example 4.24. The ring $\mathbb{Z}$ is a pre-λ-ring, with $\lambda_t(n) = (1+t)^n$.

A pre-λ-ring morphism is a ring morphism that commutes with the operations λ^d. We say that R is an *augmented* pre-λ-ring if it is equipped with a pre-λ-ring morphism $R \to \mathbb{Z}$.

Example 4.25. The canonical exterior powers $\lambda^d : SW(K) \to SW(K)$ extend to functions $GW(K) \to GW(K)$ that give $GW(K)$ a natural pre-λ-ring structure. The dimension map $\dim : GW(K) \to \mathbb{Z}$ makes $GW(K)$ an augmented pre-λ-ring.

If R is a graded ring over some abelian group G, then we say it is a *graded* pre-λ-ring if furthermore $\lambda^d(R_g) \subset R_{dg}$ for all $g \in G$ (writing G additively). A graded pre-λ-ring morphism is a pre-λ-ring morphism that is also a homogeneous map.

Example 4.26. The pre-λ-ring structure on $GW(K)$ extends to a $\mathbb{Z}/2\mathbb{Z}$-graded pre-λ-ring structure on $GW^{\pm}(K)$.

Example 4.27. If R is a G-graded pre-λ-ring and H is an abelian group, then the group ring $R[H]$ is naturally a $(G \times H)$-graded pre-λ-ring, setting $\lambda^d(x \cdot h) = \lambda^d(x) \cdot (dh)$ for all $x \in R$ and $h \in H$. Then the augmentation map $R[H] \to R$ is a morphism of G-graded pre-λ-rings.

If R is a G-graded pre-λ-ring, we say it is augmented if it has a graded pre-λ-ring $R \to \mathbb{Z}[G]$. Composing with the augmentation $\mathbb{Z}[G] \to \mathbb{Z}$ then gives a non-graded augmentation $R \to \mathbb{Z}$.

We can then state:

Proposition 4.28. *Let (A,σ) be a central simple algebra with involution of the first kind over K. The maps λ^d defined in 4.21 on $SW^{\varepsilon}(K)$ and $SW^{\varepsilon}(A,\sigma)$ extend uniquely to maps $\lambda^d : \widetilde{GW}(A,\sigma) \to \widetilde{GW}(A,\sigma)$ such that $\widetilde{GW}(A,\sigma)$ is a Γ-graded pre-λ-ring. Furthermore, the dimension map* $\mathrm{rdim} : \widetilde{GW}(A,\sigma) \to \mathbb{Z}[\Gamma]$ *(see 2.32) is a graded augmentation.*

Proof. Definition 4.21 gives functions λ^d from each component of $\widetilde{SW}(A,\sigma)$ to $\widetilde{GW}(A,\sigma)$, so they give functions λ_t from $\widetilde{SW}(A,\sigma)$ to $\Lambda(\widetilde{GW}(A,\sigma))$. Proposition 4.14 exactly shows that they are semi-group morphisms, so by the direct sum property they define a unique semi-group morphism λ_t from $\widetilde{SW}(A,\sigma)$ to $\Lambda(\widetilde{GW}(A,\sigma))$. Now the universal property of Grothendieck groups shows that this extends uniquely to a group morphism from $\widetilde{GW}(A,\sigma)$ to $\Lambda(\widetilde{GW}(A,\sigma))$.

The fact that η is a section (or equivalently that λ^1 is the identity) is clear since on each component of $\widetilde{SW}(A,\sigma)$, λ^1 is defined as the identity. So we have a pre-λ-ring structure on $\widetilde{GW}(A,\sigma)$. It preserves the grading since on each component of $\widetilde{SW}(A,\sigma)$, λ^d takes values in $SW(K)'$ if d is even, and in the component itself if d is odd.

The fact that rdim is a graded augmentation amounts to showing that if (V,h) has reduced dimension r, then $\Lambda^d(V)$ has reduced dimension $\binom{r}{d}$, which is a direct consequence of proposition 4.2. $\square$

Remark 4.29. By examples 4.26 and 4.27, $GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$ is a Γ-graded pre-λ-ring, and the observations in remark 4.23 can be reformulated as: the natural isomorphism between $\widetilde{GW}(K,\mathrm{Id})$ and $GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$ is an isomorphism of Γ-graded pre-λ-rings.

Theorem 4.30. *The functor $\widetilde{GW}$ defines a functor from $\mathbf{Br}_h(K)$ to the category of Γ-graded pre-λ-rings.*

Proof. Let $f:(B,\tau)\to(A,\sigma)$ be a morphism in $\mathbf{Br}_h(K)$. The only thing to check is that the λ-operations are compatible with the ring morphism f_*, and it is enough to check this on $SW^\varepsilon(B,\tau)$. But if $g\in SW^\varepsilon(B,\tau)$, then we have in $\mathbf{Br}_h(K)'$:

$$\begin{aligned} f\circ\lambda^d(g) &= f\circ\varphi^{(d)}_{(B,\tau)}\circ\mathrm{Alt}^d(g)\\ &= \varphi^{(d)}_{(A,\sigma)}\circ f^{\otimes d}\circ\mathrm{Alt}^d(g)\\ &= \varphi^{(d)}_{(A,\sigma)}\circ\mathrm{Alt}^d(f\circ g)\\ &= \lambda^d(f\circ g)\end{aligned}$$

using proposition 2.11 and corollary 4.17. $\square$

Remark 4.31. If $f:(B,\tau)\to(A,\sigma)$ is a morphism in $\mathbf{Br}_h(K)$, then since f_* is compatible with the λ-operations, we have $\lambda^d(f)=f_*(\lambda^d(\langle 1\rangle_\tau))$. Thus to be able to compute the exterior powers of any ε-hermitian form, we just need to be able to do the computation in the special case of diagonal forms $\langle 1\rangle_\sigma$ for any involution σ.

Example 4.32. Consider the split case $A=\mathrm{End}_K(V)$. Then the involution σ on A is adjoint to some bilinear form b on V, which is defined up to a scalar factor. Then if d is even $\lambda^d(b)$ is well-defined, while if d is odd it is only defined up to this same factor. On the other hand, the element $x_d=\lambda^d(\langle 1\rangle_\sigma)\in\widetilde{GW}(A,\sigma)$ is obviously well-defined. We can understand the relation between the two situations through functoriality. If we see the choice of b as a choice of isomorphism f_* from $\widetilde{GW}(A,\sigma)$ to $\widetilde{GW}(K,\mathrm{Id})\simeq GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$, then for d even $f_*(x_d)=\lambda^d(b)$ (in the even component) does not depend on f since $x_d\in GW(K)$, and for d odd $f_*(x_d)=\lambda^d(b)$ (in the odd component) depends on f since $x_d\in GW^\varepsilon(A,\sigma)$.

It is easy to see that the decomposition

$$A = \mathrm{Sym}(A,\sigma) \oplus \mathrm{Skew}(A,\sigma)$$

is orthogonal for the bilinear form $T_{\sigma,a,b}$ (recall proposition 2.28) for any ε-symmetric $a,b \in A^*$. We write $T^+_{\sigma,a,b}$ (resp. $T^-_{\sigma,a,b}$) for the restriction of $T_{\sigma,a,b}$ to $\mathrm{Sym}(A,\sigma)$ (resp. $\mathrm{Skew}(A,\sigma)$).

Proposition 4.33. *Let (A,σ) be an algebra with involution over K, and let $a \in A^*$ be ε-symmetric. If σ is orthogonal (resp. symplectic), then $\Lambda^2(A)$ is naturally identified with* $\mathrm{Skew}(A)$ *(resp.* $\mathrm{Sym}(A,\sigma)$*), and $\lambda^2(\langle a\rangle_\sigma)$ is isometric to $\langle\frac{1}{2}\rangle T^-_{\sigma,a,a}$ (resp. $\langle\frac{1}{2}\rangle T^+_{\sigma,a,a}$).*

Proof. The right $(A\otimes_K A)$-module $\mathrm{Alt}^2(A)$ is by definition $(1-g_A)\cdot A\otimes_K A$. So since $\varphi^{(2)}_{(A,\sigma)}$ is given by the left $(A\otimes_K A)$-module A (with twisted action), we have

$$\Lambda^2(A) = (1-g_A)\cdot A \subset A.$$

But we saw in lemma 1.3 that under the twisted action, g_A acts on A as σ if σ is orthogonal, and $-\sigma$ if σ is symplectic. So $\Lambda^2(A)$ is the subspace of A consisting of anti-symmetrized elements if σ is orthogonal, and of symmetrized elements if σ is symplectic. Since 2 in invertible in K, this means that $\Lambda^2(A) = \mathrm{Skew}(A,\sigma)$ if σ is orthogonal, and $\Lambda^2(A) = \mathrm{Sym}(A,\sigma)$ if σ is symplectic (see for instance the discussion in [11, 2.A]).

Since we know from remark 4.13 that the restriction of $\langle a\rangle^2_\sigma$ to $\Lambda^2(A)$ is $\langle 2\rangle\lambda^2(\langle a\rangle_\sigma)$, and from proposition 2.28 that $\langle a\rangle^2_\sigma = T_{\sigma,a,a}$, we may conclude. □

4.5 Norm and determinant

The construction of the exterior powers $\lambda^d(A)$ of an algebra A of degree n has a very interesting special case, namely when $d = n$. Indeed, in this case $\lambda^n(A)$ has degree 1, so it is canonically isomorphic to K. This means that $\mathrm{Alt}^n(A)$ gives an explicit Brauer-equivalence between $A^{\otimes n}$ and K, which is a possible way to prove without using cohomology that the Brauer group is a torsion group, and that the exponent of A divides its degree (it is used for instance in [20], and the idea is attributed to Tamagawa).

In the split case, this is how one defines the determinant of an endomorphism: if $f \in \mathrm{End}_K(V)$, it induces $\Lambda^n(f) \in \mathrm{End}_K(\Lambda^n(V))$, which is a homothety since $\Lambda^n(V)$ has dimension 1, and the corresponding scalar $\det(f) \in K$ is called the *determinant* of f. We can imitate this definition in the general case: let A be a central simple algebra over K and $d \in \mathbb{N}$; there is an obvious action of $\mathfrak{S}_d$ on $A^{\otimes d}$, and we write $\mathrm{Sym}^d(A) \subset A^{\otimes d}$ for the subalgebra of fixed points under this action. Then any $x \in \mathrm{Sym}^d(A)$ commutes with $s_{d,A}$, so the application $a \mapsto xa$ on $A^{\otimes d}$ stabilizes $\mathrm{Alt}^d(A) = s_{d,A}A^{\otimes d}$. This defines a canonical map

$$\mathrm{Sym}^d(A) \longrightarrow \lambda^d(A).$$

The composition with the natural map $A \to \mathrm{Sym}^d(A)$ given by $a \mapsto a\otimes\cdots\otimes a$ then defines a map

$$\lambda^d : A \longrightarrow \mathrm{Sym}^d(A) \longrightarrow \lambda^d(A).$$

Clearly, λ^d is compatible with scalar extension, and when $A = \mathrm{End}_K(V)$ is split, this is the usual map from $\mathrm{End}_K(V)$ to $\mathrm{End}_K(\Lambda^d(V))$ induced by the functoriality of exterior powers.

Proposition 4.34. *Let A be a central simple algebra over K, of degree n. Then the map $\lambda^n : A \to K$ is the reduced norm* Nrd_A.

Proof. Since the reduced norm is usually defined by descent, and since the maps λ^d are compatible with base change, we can check this when A is split. But then as we discussed above λ^n is the determinant map, which is the reduced norm in the split case. □

Remark 4.35. This may be taken as a definition of the reduced norm, which avoids the use of Galois descent. In more concrete terms, this definition can be rephrased as: $a^{\otimes n} s_{d,n} = \mathrm{Nrd}_A(a) s_{d,n}$.

There is also a classical notion of determinant for algebras with involution, which can be defined as a descent of the determinant of a bilinear form (a definition that avoids any splitting argument is given in [11, 7.2]). However, it is only defined for algebras of even degree; the reason is clear if we use the descent definition: when the degree n of A is odd, then A is already split, but the bilinear form corresponding to the involution is only well-defined up to a scalar factor, so its determinant is not well-defined (this problem does not exist in even degree since the determinant of quadratic forms of even dimension is a similitude invariant). We suggest a slightly more general definition that works in arbitrary degree.

Definition 4.36. *Let (A, σ) be a central simple algebra with involution of the first kind over K, and let (V, h) be a ε-hermitian module over (A, σ). Then the determinant of (V, h) (often called the determinant of h) is*

$$\det(V, h) = \det(h) = \lambda^n(V, h) \in \widetilde{GW}(A, \sigma),$$

with $n = \mathrm{rdim}_A(V)$. The determinant of (A, σ) is

$$\det(A, \sigma) = \det(\sigma) = \det(\langle 1 \rangle_\sigma) \in \widetilde{GW}(A, \sigma).$$

Remark 4.37. If σ is symplectic, then usually there is no particular notion of $\det(\sigma)$, and indeed with our definition we always have $\det(\sigma) = \langle 1 \rangle \in GW(K)$.

Remark 4.38. When the degree of A is even and σ is orthogonal, then with this definition $\det(\sigma) \in GW(K)$ is a 1-dimensional quadratic form, so it has the form $\langle d \rangle$ for some $d \in K^*$, and by construction the square class of d is the usual determinant of σ as defined in [11] (we can check this in the split case). We usually identify the two definitions in this case (so we identify a square class wth the 1-dimensional form it defines).

Remark 4.39. When $\deg(A) = n$ is odd, then A is split and σ is orthogonal, and $\det(\sigma) \in GW(A, \sigma)$. The choice of some bilinear form b such that $\sigma = \sigma_b$ gives an isomorphism $\widetilde{GW}(A, \sigma) \simeq GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$ that sends $\det(\sigma)$ to $\langle \det(b) \rangle$ in the odd component. So while $\det(\sigma)$ is well-defined in $\widetilde{GW}(A, \sigma)$, its interpretation as a square class indeed depends on the choice of some b, which is the classical obstruction.

In the classical theory of bilinear forms, the pairing

$$\Lambda^p(V) \otimes_K \Lambda^q(V) \longrightarrow \Lambda^{p+q}(V)$$

induces a sort of duality when $p+q=n=\dim_K(V)$. Indeed, since $\Lambda^n(V)$ has dimension 1, by choosing a basis vector in $\Lambda^n(V)$ we get $\Lambda^p(V)\simeq\Lambda^q(V)^\vee$. If we restrain from choosing a basis of $\Lambda^n(V)$, then we get an isomorphism

$$\Lambda^p(V)\simeq\Lambda^n(V)\otimes_K\Lambda^q(V)^\vee\simeq\Lambda^n(V)\otimes_K\Lambda^q(V^\vee),$$

using that $\Lambda^d(V)^\vee\simeq\Lambda^d(V^\vee)$ for any $d\in\mathbb{N}$. Now if V is equipped with a non-degenerated bilinear form b, then using the identification $V\simeq V^\vee$ given by b, the isomorphism gives an isometry

$$(\Lambda^p(V),\lambda^p(b))\simeq(\Lambda^n(V),\lambda^n(b))\otimes_K(\Lambda^q(V),\lambda^q(b)).$$

So in the end we get an equality $\lambda^p(b)=\det(b)\lambda^q(b)$ in $GW(K)$ (or possibly in $GW^\pm(K)$ is b is anti-symmetric). We want to generalize this duality formula in the non-split case. First we show a general lemma on modules:

Lemma 4.40. *Let A and B be K-algebras, and let U, V and W be right modules over, respectively, A, B and $A\otimes_K B$. Then $U\otimes_K B$ is a right module over $A\otimes_K B\otimes_K B^{op}$, where B has its standard structure of $(B\otimes_K B^{op})$-module (here on the right). Likewise,* $\mathrm{Hom}_K(V,W)$ *is a module over $A\otimes_K B\otimes_K B^{op}$ through the action of $A\otimes_K B$ on W and the action of B on V.*

There is a natural isomorphism

$$\begin{array}{ccc}\mathrm{Hom}_{A\otimes_K B}(U\otimes_K V,W) & \xrightarrow{\sim} & \mathrm{Hom}_{A\otimes_K B\otimes_K B^{op}}(U\otimes_K B,\mathrm{Hom}_K(V,W)) \\ f & \longmapsto & (u\otimes b\mapsto(v\mapsto f(u\otimes vb)))\,. \end{array}$$

Proof. A simple calculation shows that all actions are indeed respected, and that the inverse is given by $g\mapsto(u\otimes v\mapsto g(u\otimes 1)(v))$. □

Now if A and B are central simple algebras over K and V is a Morita B-A-bimodule, recall that $V^\vee$ is a A-B-bimodule, and we write V^{-1} for the same module seen as a B^{op}-A^{op}-bimodule.

Lemma 4.41. *There is a natural isomorphism of A-B-bimodule $V^\vee\simeq\mathrm{Hom}_K(V,K)$, given by either*

$$\begin{array}{ccc}\mathrm{Hom}_A(V,A) & \xrightarrow{\sim} & \mathrm{Hom}_K(V,K) \\ f & \longmapsto & \mathrm{Trd}_A\circ f\end{array}$$

or

$$\begin{array}{ccc}\mathrm{Hom}_B(V,B) & \xrightarrow{\sim} & \mathrm{Hom}_K(V,K) \\ f & \longmapsto & \mathrm{Trd}_B\circ f\end{array}$$

Proof. It is an easy verification that the maps are well-defined and are bimodule morphisms. To see that they are bijective it suffices to check that they are injective, since the K-dimensions are the same. But if f is in the kernel, it means that for any $v\in V$, we have for all $a\in A$:

$$\mathrm{Trd}_A(f(v)a)=\mathrm{Trd}_A(f(va))=0,$$

so $f(v)=0$ since the trace form is non-degenerated.

Recall that the canonical identification $\mathrm{Hom}_A(V,A)\simeq\mathrm{Hom}_B(V,B)$ is given by

$$f(x)y=xf'(y)$$

for all $x, y \in V$, where $f \in \mathrm{Hom}_B(V, B)$ corresponds to $f' \in \mathrm{Hom}_A(V, A)$. Then to establish that the two isomorphisms correspond to each other through this identification, we have to show that $\mathrm{Trd}_B(f(x)) = \mathrm{Trd}_A(f'(x))$ for all $x \in V$. It is enough to check this in the split case; then $A \simeq \mathrm{End}_K(U) \simeq U \otimes_K U^\vee$, $B \simeq \mathrm{End}_K(W) \simeq W \otimes_K W^\vee$, and $V \simeq W \otimes_K U^\vee$. The reduced trace of A is given by $(u \otimes \varphi \mapsto \varphi(u))$, and likewise for B. We have an identification $V^\vee \simeq U \otimes_K W^\vee$, such that if f corresponds to $u \otimes \varphi$, then for $x = w \otimes \psi \in V$, $f(x) = \psi(u) w \otimes \varphi$ and $f'(x) = \varphi(w) u \otimes \psi$. In the end:

$$\mathrm{Trd}_B(f(x)) = \psi(u)\varphi(w) = \mathrm{Trd}_A(f'(x)). \qquad \square$$

Proposition 4.42. *Let A be a central simple algebra over K, let V be a right A-module of reduced dimension n, and let $p, q \in \mathbb{N}$ such that $p + q = n$. There is a canonical isomorphism of $(A^{\otimes n} \otimes_K (A^{op})^{\otimes q})$-modules*

$$\Phi_{p,q}(V) : \mathrm{Alt}^p(V) \otimes_K A^{\otimes q} \xrightarrow{\sim} \mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V)^{-1}.$$

Proof. If we apply the correspondance of lemma 4.40 to the shuffle map

$$\mathrm{Alt}^p(V) \otimes_K \mathrm{Alt}^q(V) \longrightarrow \mathrm{Alt}^n(V),$$

we get a morphism of $(A^{\otimes n} \otimes_K (A^{op})^{\otimes q})$-modules

$$\mathrm{Alt}^p(V) \otimes_K A^{\otimes q} \longrightarrow \mathrm{Hom}_K(\mathrm{Alt}^q(V), \mathrm{Alt}^n(V)),$$

and applying lemma 4.41 we get a morphism

$$\mathrm{Alt}^p(V) \otimes_K A^{\otimes q} \longrightarrow \mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V)^{-1}.$$

To prove that this is an isomorphism, it is enough to check it in the split case. Using the same notations as in the proof of lemma 4.41, this becomes a morphism

$$\Lambda^p(W) \otimes_K U^{\otimes p} \otimes_K U^{\otimes q} \otimes_K (U^\vee)^{\otimes q} \longrightarrow \Lambda^n(W) \otimes_K U^{\otimes n} \otimes_K (\Lambda^q(W))^\vee \otimes_K (U^\vee)^{\otimes q},$$

and it is a lengthy but simple verification to see that this is the tensor product of the usual isomorphism

$$\Lambda^p(W) \xrightarrow{\sim} \Lambda^n(W) \otimes_K (\Lambda^q(W))^\vee$$

with $U^{\otimes n} \otimes_K (U^\vee)^{\otimes q}$. $\square$

Remark 4.43. The module isomorphism $\Phi_{p,q}(V)$ induces an isomorphism between the endomorphism algebras of either side: this gives a canonical isomorphism $\lambda^p(A) \simeq \lambda^q(A)^{op}$. This is the isomorphism alluded to in [11, exercise II.12]. In particular, when $n = 2m$, this defines an isomorphism $\lambda^m(A) \simeq \lambda^m(A)^{op}$ which corresponds to the so-called canonical involution on $\lambda^m(A)$ (see [11, §10.B]).

If there is an involution σ on A, then the right $(A^{op})^{\otimes q}$-module $\mathrm{Alt}^q(V)^{-1}$ corresponds to the right $A^{\otimes q}$-module $\mathrm{Alt}^q(V)^*$, and the right $A^{\otimes q} \otimes_K (A^{op})^{\otimes q}$-module $A^{\otimes q}$ corresponds to the right $A^{\otimes 2q}$-module $\overline{A^{\otimes q}}$, where we see $A^{\otimes q}$ as a left $A^{\otimes 2q}$-module through the twisted sandwich action. So $\Phi_{p,q}(V)$ becomes an isomorphism of right $A^{\otimes n+q}$-modules

$$\mathrm{Alt}^p(V) \otimes_K \overline{A^{\otimes q}} \xrightarrow{\sim} \mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V)^*.$$

If furthermore we have a ε-hermitian form h on V, then this induces an isomorphism

$$\mathrm{Alt}^p(V) \otimes_K \overline{A^{\otimes q}} \xrightarrow{\sim} \mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V). \tag{24}$$

In addition, we can consider the inverse of $\varphi^{(2q)}_{(A,\sigma)}$ in $\mathbf{Br}_h(K)$ (recall 1.17):

$$\overline{\varphi^{(2q)}_{(A,\sigma)}} : (K, \mathrm{Id}) \longrightarrow (A^{\otimes 2q}, \sigma^{\otimes 2q})$$

which is a hermitian form on $\overline{A^{\otimes q}}$. So each module in (24) carries a hermitian form.

Proposition 4.44. *Let (A,σ) be an algebra with involution over K, let (V,h) be a ε-hermitian module over (A,σ) of reduced dimension n, and let $p,q \in \mathbb{N}$ such that $p+q=n$. Then the $A^{\otimes n+q}$-module isomorphism (24) induced by $\Phi_{p,q}(V)$ is an isometry*

$$\mathrm{Alt}^p(h) \otimes \overline{\varphi^{(2q)}_{(A,\sigma)}} \xrightarrow{\sim} \mathrm{Alt}^n(h) \otimes \mathrm{Alt}^q(h).$$

Proof. To check that this gives an isometry, we can once again reduce to the split case, and we use the same notations as in the proof of proposition 4.42. We have bilinear forms b on U and c on W such that the following diagram in commutative in $\mathbf{Br}_h(K)$:

$$\begin{array}{ccc} (B,\tau) & \xrightarrow{(V,h)} & (A,\sigma) \\ & {\scriptstyle (W,c)}\searrow \quad \swarrow{\scriptstyle (U,b)} & \\ & (K,\mathrm{Id}). & \end{array}$$

They induce identifications $U^\vee \simeq U$ and $W^\vee \simeq W$ given by $\hat{b}$ and $\hat{c}$, so in particular $A \simeq U \otimes_K U$ and $V \simeq W \otimes_K U$. Then the map (24) becomes

$$\Lambda^p(W) \otimes_K U^{\otimes p} \otimes_K U^{\otimes q} \otimes_K U^{\otimes q} \longrightarrow \Lambda^n(W) \otimes_K U^{\otimes n} \otimes_K \Lambda^q(W) \otimes_K U^{\otimes q},$$

and we can check that the hermitian forms which we have to show are isometric are, on the left:

$$(x \otimes u_1 \otimes u_2 \otimes u_3, y \otimes v_1 \otimes v_2 \otimes v_3) \mapsto \lambda^p(c)(x,y) \cdot (u_1 \otimes v_1) \otimes (u_2 \otimes v_2) \otimes (u_3 \otimes v_3)$$

with and on the right:

$$(x \otimes u \otimes y \otimes v, x' \otimes u' \otimes y' \otimes v') \mapsto \lambda^n(c)(x,x') \cdot \lambda^q(c)(y,y') \cdot (u \otimes u') \otimes (v \otimes v').$$

Thus this is the tensor product of the usual isometry

$$\lambda^p(c) \simeq \lambda^n(c)\lambda^q(c)$$

with $U^{\otimes n+q}$. ∎

We can finally prove:

Corollary 4.45. *Let (A,σ) be an algebra with involution over K, let (V,h) be a ε-hermitian module over (A,σ) of reduced dimension n, and let $p,q \in \mathbb{N}$ such that $p+q=n$. Then we have in $\widetilde{GW}(A,\sigma)$:*

$$\lambda^p(h) = \det(h) \cdot \lambda^q(h).$$

Proof. By definition, we have

$$\lambda^p(h) = \varphi^{(n+q)}_{(A,\sigma)} \circ \left(\mathrm{Alt}^p(h) \otimes \overline{\varphi^{(2q)}_{(A,\sigma)}}\right)$$

and

$$\det(h) \cdot \lambda^q(h) = \varphi^{(n+q)}_{(A,\sigma)} \circ (\mathrm{Alt}^n(h) \otimes \mathrm{Alt}^q(h))$$

so this follows directly from the previous proposition. □

4.6 Some open questions and partial answers

We would like to point out some natural questions that arise from the study of $\widetilde{GW}(A,\sigma)$ as a pre-λ-ring, and to which we cannot yet give a full answer.

λ-rings

The most obvious question is:

Question 1: Is $\widetilde{GW}(A,\sigma)$ a λ-ring ?

We recall what it means for a pre-λ-ring to be a λ-ring (see [24] or [25] for more details). If R is any ring, there is actually a natural pre-λ-ring structure on $\Lambda(R)$, and R is a λ-ring if the group morphism $\lambda_t : R \to \Lambda(R)$ is actually a pre-λ-ring morphism. This is the case of most pre-λ-rings that naturally arise in practice. Concretely, there are certain universal polynomials $P_n(x_1, \dots, x_n; y_1, \dots, y_n)$ and $P_{n,m}(x_1, \dots, x_{nm})$ with coefficients in $\mathbb{Z}$, for $n, m \in \mathbb{N}^*$, such that a pre-λ-ring is a λ-ring iff for any $x, y \in R$, we have

$$\lambda^n(xy) = P_n(\lambda^1(x), \dots, \lambda^n(x), \lambda^1(y), \dots, \lambda^n(y)) \tag{25}$$

$$\lambda^n(\lambda^m(x)) = P_{n,m}(\lambda^1(x), \dots, \lambda^{nm}(x)). \tag{26}$$

Remark 4.46. Since the λ-structure is functorial with respect to $\mathbf{Br}_h(K)$, we see that the fact that $\widetilde{GW}(A,\sigma)$ is a λ-ring only depends on the Brauer class of A.

If R is any pre-λ-ring, and $x \in R$, it is called finite-dimensional if $\lambda_t(x) \in R[[t]]$ is actually a polynomial. Its *dimension* is then the degree of $\lambda_t(x)$. A λ-ring enjoys the following property, which we call the *dimension property*: the unit $1 \in R$ is 1-dimensional, and a product of 1-dimensional elements is 1-dimensional. Our first hint that $\widetilde{GW}(A,\sigma)$ might just be a λ-ring is:

Proposition 4.47. *Let (A,σ) be an algebra with involution over K. Then $\widetilde{GW}(A,\sigma)$, with its canonical structure of pre-λ-ring, satisfies the dimension property.*

We will actually show something a little stronger. We take the following definition from [25]: if R is an augmented pre-λ-ring (see the discussion before proposition 4.28), with augmentation $d : R \to \mathbb{Z}$, a *positive structure* on R is a subset $R_{>0}$, whose elements are called *positive*, such that:

- $0 \notin R_{>0}$;

- $R_{>0} \cup \{0\}$ is stable under sums, products and operations λ^d;
- any element of R is a difference of two positive elements;
- any $x \in R_{>0}$ is finite-dimensional, with dimension $d(x)$, and $\lambda^{d(x)} \in R^*$;
- the 1-dimensional positive elements form a subgroup of R^*.

The 1-dimensional positive elements of R are called *line elements*. If R is a *graded* pre-λ-ring, then a *homogeneous* positive structure is a positive structure which is a homogeneous subset of R (so the homogeneous components of positive elements are positive), and in this case we just have to verify the conditions for homogeneous elements. It is then easy to show:

Lemma 4.48. *Let R be a pre-λ-ring with a positive structure. Then all 1-dimensional elements of R are line elements. In particular, R satisfies the dimension property.*

Proof. Let $x \in R$ be 1-dimensional, and write $x = x_1 - x_2$ with $x_i \in R_{>0}$, and $d_i = d(x_i) \in \mathbb{N}^*$. Then $\lambda_t(x)\lambda_t(x_2) = \lambda_t(x_1)$, so $x\lambda^{d_2}(x_2) = \lambda^{d_1}(x_1)$. Now $\lambda^{d_i}(x_i)$ is a line element by hypothesis, so x is also a line element. □

We have to be careful that if $x \in R$ is positive, then $d(x)$ is its dimension, but for a general x, we can have $d(x) = 1$ without x having finite dimension. Recall from proposition 4.28 that $\widetilde{GW}(A, \sigma)$ is augmented.

Lemma 4.49. *Let (A, σ) be an algebra with involution over K. Then $\widetilde{SW}(A, \sigma)$ defines a homogeneous positive structure on $\widetilde{GW}(A, \sigma)$.*

Proof. Everything is more or less obvious, except maybe the invertibility properties. They follow from the fact that if x is positive and homogeneous of dimension n then $\lambda^n(x) = \det(x)$ so it is invertible, and if x is a line element its multiplicative inverse is itself. □

Lemmas 4.48 and 4.49 prove proposition 4.47. One of the main interest of positive structures is that to show that R is a λ-ring, it is enough to establish conditions (25) and (26) when x and y are positive, and actually it is enough to consider a subset S of $R_{>0}$ such that every positive element is a sum of elements in S (see [25, 3.1]). In particular, if we have a homogeneous positive structure, then it is enough to consider homogeneous positive elements. The polynomials P_n and $P_{n,m}$ are defined in such a way that any sum of line elements satisfies conditions (25) and (26), which is the usual way to show that $GW(K)$ is a λ-ring (it is more or less the method used in [16]). Furthermore, Zibrowius shows the following lemma:

Lemma 4.50 ([25], lemma 3.4)**.** *Let R be a pre-λ-ring with a positive structure. Then two positive elements x and y of dimension at most 2 satisfy conditions (25) and (26) iff they satisfy (25) for $n = 2, 3, 4$, which is to say:*

(i) $\lambda^2(xy) = x^2\lambda^2(y) + \lambda^2(x)y^2 - 2\lambda^2(x)\lambda^2(y)$;

(ii) $\lambda^3(xy) = xy\lambda^2(x)\lambda^2(y)$;

(iii) $\lambda^4(xy) = (\lambda^2(x))^2(\lambda^2(y))^2$.

This allows us to show:

Proposition 4.51. *Let (A, σ) be a split algebra with involution over K. Then $\widetilde{GW}(A, \sigma)$ is a λ-ring.*

Proof. We are reduced to the case $(A, \sigma) = (K, \mathrm{Id})$ by remark 4.46. Since the positive elements are additively generated by 1-dimensional diagonal forms (in the even and the odd components) and the anti-symmetric hyperbolic plane (in the even and the odd components), it is enough to check the conditions for these elements, and since they all have dimension at most 2, we can use lemma 4.50. The formulas in the lemma are immediate if x or y has the form $\langle a \rangle$ (in either component), so only the case of two anti-symmetric planes remains. It is then an immediate computation, using that $x^2 = y^2 = xy = 2\mathcal{H}$ (where $\mathcal{H}$ is the symmetric hyperbolic plane), and $\lambda^2(x) = \lambda^2(y) = \langle 1 \rangle$. □

Remark 4.52. As far as we know, even the fact that $GW^{\pm}(K)$ is a λ-ring had not appeared explicitly in the literature, most likely because few people care about $GW^{\pm}(K)$.

We can also prove:

Proposition 4.53. *Let (A, σ) be an algebra with involution over K of index 2. Then $\widetilde{GW}(A, \sigma)$ is a λ-ring.*

Proof. We are reduced to the case where $(A, \sigma) = (Q, \gamma)$ is a quaternion division algebra with its canonical involution. Using lemma 4.50, we have to show those three identities when x and y are of the form $\langle a \rangle$ for $a \in K^*$, $\mathcal{H}_{-1}$ (the anti-symmetric hyperbolic plane), $\langle a \rangle_\gamma$ with $a \in K^*$, or $\langle z \rangle_\gamma$ with z a non-zero pure quaternion. The cases where x or y is $\langle a \rangle$ are immediate, and the case $x = y = \mathcal{H}_{-1}$ follows from proposition 4.51. Also note that we can easily replace $\langle a \rangle_\gamma$ by $\langle 1 \rangle_\gamma$ for any $a \in K^*$.

Note that when x and y are both of dimension 2, then $\det(xy) = \langle 1 \rangle$, since according to proposition 3.22, in this case xy is either hyperbolic (of reduced dimension 4) or a multiple of a 2-fold Pfister form. This remark takes care of equation (iii) since it can be rephrased as $\det(xy) = \det(x)^2 \det(y)^2$, and of equation (ii) since using corollary 4.45, we have $\lambda^3(xy) = \det(xy)xy$, $\lambda^2(x)x = \det(x)x = x$ and likewise $\lambda^2(y)y = \det(y)y = y$.

It remains to show equation (i) when x any y are both of dimension 2, and are not both $\mathcal{H}_{-1}$. Thus we have five cases to check. Note that since in each case the dimensions coincide on both sides, we may as well prove that the equality holds in $W(K)$ instead of $GW(K)$.

We first recall two formulas that are easy to establish, for any $a, b, c \in K^*$:

$$\lambda^2(\langle\langle a, b \rangle\rangle) = 2(\langle\langle a, b \rangle\rangle - 1) \tag{27}$$

$$\langle\langle a, c \rangle\rangle + \langle\langle b, c \rangle\rangle = \langle\langle ab, c \rangle\rangle + \langle\langle a, b, c \rangle\rangle \in W(K). \tag{28}$$

If $x = y = \langle 1 \rangle_\gamma$, the equation is:

$$\lambda^2(\langle 2 \rangle n_Q) = \langle 2 \rangle n_Q + \langle 2 \rangle n_Q - 2$$

which follows from (27).

If $x = \mathcal{H}_{-1}$ and $y = \langle 1\rangle_\gamma$, then xy is hyperbolic. Let us take z_1, z_2 pure quaternions that anti-commute. Then $xy = \langle z_1, -z_1\rangle$, and $n_Q = \langle\langle z_1^2, z_2^2\rangle\rangle$. Equation (i) then becomes:

$$\lambda^2(\langle z_1, -z_1\rangle_\gamma) = 2\mathcal{H}\cdot\langle 1\rangle + \langle 1\rangle\cdot\langle 2\rangle n_Q - 2\langle 1\rangle\langle 1\rangle.$$

But

$$\begin{aligned}\lambda^2(\langle z_1, -z_1\rangle_\gamma) &= \lambda^2(\langle z_1\rangle_\gamma) + \langle z_1\rangle_\gamma\langle -z_1\rangle_\gamma + \lambda^2(\langle -z_1\rangle_\gamma)\\ &= 2\langle -z_1^2\rangle + \langle 2z_1^2\rangle\langle\langle z_1^2, -z_2^2\rangle\rangle\end{aligned}$$

so we can check that the equation can be rearranged as

$$\langle\langle -1, z_1^2\rangle\rangle = \langle 2\rangle\langle\langle z_1^2, z_2^2\rangle\rangle + \langle 2\rangle\langle\langle z_1^1, -z_2^2\rangle\rangle \tag{29}$$

which follows from (28).

If $x = \mathcal{H}_{-1}$ and $y = \langle z_1\rangle_\gamma$, then xy is hyperbolic, so $xy = \langle 1, -1\rangle_\gamma$. Let us choose some z_2 that anti-commutes with z_1. The equation then becomes:

$$\lambda^2(\langle 1, -1\rangle_\gamma) = 2\mathcal{H}\cdot\langle -z_1^2\rangle + \langle -2z_1^2\rangle\langle\langle z_1^2, z_2^2\rangle\rangle - 2\langle -z_1^2\rangle.$$

But since $\lambda^2(\langle 1, -1\rangle_\gamma) = 2 - \langle 2\rangle n_Q$, this can be rearranged to give the same equation as (29).

If $x = \langle z\rangle_\gamma$ and $y = \langle 1\rangle_\gamma$, then taking some z_2 that anti-commutes with z, the equation becomes

$$\lambda^2(2\mathcal{H}_{-1}) = \langle -2z^2\rangle\langle\langle z^2, z_2^2\rangle\rangle + \langle\langle -2z^2\rangle\rangle n_Q - 2\langle -z^2\rangle.$$

But since $\lambda^2(2\mathcal{H}_{-1}) = 2(\mathcal{H}+1)$, this can again be rearranged as (29).

Fincally, if $x = \langle z_1\rangle\gamma$ and $y = \langle z_2\rangle\gamma$, then we choose z_0 that anti-commutes with both z_1 and z_2. The equation becomes:

$$\lambda^2(\langle -\operatorname{Trd}_Q(z_1z_2)\rangle\varphi_{z_1,z_2}) = \langle 2z_1^2z_2^2\rangle\langle\langle z_1^2, -z_0^2\rangle\rangle + \langle 2z_2^2z_1^2\rangle\langle\langle z_2^2, -z_0^2\rangle\rangle - 2\langle -z_1^2\rangle\langle -z_2^2\rangle,$$

which using (27) gives:

$$2\varphi_{z_1,z_2} = \langle 2z_1^2z_2^2\rangle(\langle\langle z_1^2, -z_0^2\rangle\rangle + \langle\langle z_2^2, -z_0^2\rangle\rangle) + \langle\langle -1, z_1^2z_2^2\rangle\rangle.$$

Now $\varphi_{z_1,z_2} = \langle\langle z_1^2, z_2^2z_0^2\rangle\rangle = \langle\langle z_2^2, z_2^1z_0^2\rangle\rangle$ so in particular $2\varphi_{z_1,z_2}$ represents $2z_1^2z_2^2$; also, $\langle\langle -1, z_1^2z_2^2\rangle\rangle$ represents $-2z_1^2z_2^2$, so in the end we can rewrite the equation as:

$$\langle\langle -1, z_1^2, z_2^2z_0^2\rangle\rangle + \langle\langle -1, z_1^2z_2^2\rangle\rangle = \langle\langle z_1^2, -z_0^2\rangle\rangle + \langle\langle z_2^2, -z_0^2\rangle\rangle.$$

Using (28), this means:

$$\langle\langle -1, z_1^2, z_2^2z_0^2\rangle\rangle + \langle\langle -1, z_1^2z_2^2\rangle\rangle = \langle\langle -z_0^2, z_1^2z_2^2\rangle\rangle + \langle\langle -z_0^2, z_1^2, z_2^2\rangle\rangle.$$

We can check using $\langle\langle z_1^2, z_0^2\rangle\rangle = \langle\langle z_2^2, z_0^2\rangle\rangle$ that the 2-fold and 3-fold Pfister forms on either side are the same, which concludes. □

Those are as of now the only cases of question 1 that we know how to solve. Note that if it had a positive answer in general, this would have direct consequences for (twisted) involution trace forms, since it would give universal formulas for $\lambda^d(T_{\sigma,a}) = \lambda^d(\langle 1\rangle_\sigma\langle a\rangle_\sigma)$ in terms of the $\lambda^i(\langle 1\rangle_\sigma)$ and $\lambda^j(\langle a\rangle_\sigma)$. For instance, the formula for $\lambda^2(xy)$ in a λ-ring yields that if $\widetilde{GW}(A,\sigma)$ is a λ-ring, then (assuming for simplicity that σ is orthogonal and $a \in A^*$ is symmetric):

$$\lambda^2(T_{\sigma,a}) = T_\sigma T^-_{\sigma_a} + T^-_\sigma T_{\sigma_a} - 2T^-_\sigma T^-_{\sigma_a}.$$

Similarly, from the formula for $\lambda^2(\lambda^2(x))$, we would get

$$\lambda^2(T^-_\sigma) = \langle 1\rangle_\sigma \lambda^3(\langle 1\rangle_\sigma) - \lambda^4(\langle 1\rangle_\sigma),$$

so if for instance $\deg(A) = 4$ this would mean

$$\lambda^2(T^-_\sigma) = \det(\sigma)(T_\sigma - \langle 1\rangle).$$

Natural operations on $\widetilde{GW}(A,\sigma)$

If we consider not only an individual $GW(K)$ but the family of all $GW(L)$ for all field extensions L/K then Serre shows in [8] that the λ-operations play a very special role. Indeed, let $\mathrm{Field}_{/K}$ be the category of field extensions of K. Then W is a functor from $\mathrm{Field}_{/K}$ to the category of $W(K)$-algebras. For any $r \in \mathbb{N}^*$, if we write $\mathrm{Quad}_r(L)$ for the set of non-degenerated quadratic forms of dimension r over L, then Quad_r defines a functor from $\mathrm{Field}_{/K}$ to the category of sets. A natural transformation from Quad_r to W (seen as a set-valued functor) is called a Witt invariant of Quad_n. The set $\mathrm{Inv}(\mathrm{Quad}_r, W)$ of such invariants is naturally a $W(K)$-algebra, and Serre shows that it is actually a free module with basis $(\lambda^d)_{d\leqslant r}$.

Now let (A,σ) be an algebra with involution over K. Then we have a functor $\widetilde{W}(A,\sigma)$ from $\mathrm{Field}_{/K}$ to the category of $\widetilde{W}(A,\sigma)$-algebras, sending L/K to $\widetilde{W}(A_L,\sigma_L)$. Furthermore, we can define a functor $\mathrm{Herm}_{r,\varepsilon}(A,\sigma)$ for any $r \in \mathbb{N}^*$ such that $\mathrm{Herm}_r(A,\sigma)(L)$ is the set of elements in $SW_\varepsilon(A_L,\sigma_L)$ with reduced dimension r. Then we have a $\widetilde{W}(A,\sigma)$-algebra of invariants $\mathrm{Inv}(\mathrm{Herm}_{r,\varepsilon}(A,\sigma), \widetilde{W}(A,\sigma))$.

Question 2: Is $\mathrm{Inv}(\mathrm{Herm}_{r,\varepsilon}(A,\sigma), \widetilde{W}(A,\sigma))$ generated by $(\lambda^d)_{d\leqslant r}$ as a $\widetilde{W}(A,\sigma)$-module ?

We do not have as many hints that the answer may be positive as we did for question 1, but we still conjecture it should be. In the case of quadratic forms, Serre proves his result using methods of specialization with respect to valuations; it is natural to expect that a similar theory for mixed Witt rings may be useful to address question 2. Note at least that that the answer to question 2 only depends on the Brauer class of A (for fixed r and ε), and that the answer is positive for split algebras, as a direct consequence of Serre's theorem.

Remark 4.54. It is not true in general when A is not split that the λ^d are independent invariants, even over $W(K)$. Indeed, if we consider the case of a division quaternion algebra Q with its canonical involution γ, then it is not hard to see that $n_Q(1-\lambda^2) = 0$ and $n_Q\lambda^1 = 0$ where n_Q is the norm form.

Pre-λ-rings of linear type

We may first want to study a weaker version of question 2, that can be asked about a single $\widetilde{GW}(A,\sigma)$ (instead of the family of those algebras obtained by field extensions). If question 2 has a positive answer, then it is certainly true that for any $r \in \mathbb{N}^*$, any $\varepsilon = \pm 1$ and any $i, j \leqslant r$, there exist elements $N_\varepsilon^r(i,j,d) \in \widetilde{GW}(A,\sigma)$ for all integers $d \leqslant 0$ such that

$$\lambda^i(x)\lambda^j(x) = \sum_{d=0}^{r} N_\varepsilon^r(i,j,d)\lambda^d(x) \tag{30}$$

for all $x \in \mathrm{Herm}_{r,\varepsilon}(A,\sigma)(K)$. This is because $x \mapsto \lambda^i(x)\lambda^j(x)$ is obviously an invariant since it is compatible with field extensions.

For instance, for quadratic forms it is not too difficult to see that we can take $N^r(i,j,d) = \binom{r-d}{\frac{i+j-d}{2}}\binom{d}{\frac{d+i-j}{2}}$ (which is understood to be 0 when $i+j$ and d do not have the same parity).

In general, let R be a Γ-graded pre-λ-ring. We say that R is of linear type if it admits a homogeneous positive structure $R_{>0}$ such that for every $r \in \mathbb{N}^*$, every $g \in \Gamma$, and every $i, j \leqslant r$, there are elements $N_g^r(i,j,d) \in R$ such that the formula (30) holds for any positive homogeneous x of dimension r in R_g. The reason for this name is that in that case, for any polynomial $P \in R[x_1, \dots, x_r]$, there exists $Q \in R[x_1, \dots, x_r]$ of degree 1 (depending on P, r and g) such that

$$P(\lambda^1(x), \dots, \lambda^r(x)) = Q(\lambda^1(x), \dots, \lambda^r(x))$$

for any positive homogeneous x of dimension r in R_g. In other words, any polynomial expression in the $\lambda^i(x)$ is equal to a linear one.

Note that this is not true in general even for λ-rings, since in general the operations λ^i are algebraically independent (this is the case for free λ-rings).

Question 3: Is $\widetilde{GW}(A,\sigma)$ of linear type, for it canonical homogeneous positive structure given by $\widetilde{SW}(A,\sigma)$?

As we explained, a positive answer to question 2 implies a positive answer for question 3. In particular, it has a positive answer for split algebras, since Serre's theorem takes care of quadratic forms (in the even or the odd component), and the case of anti-symmetric forms is even easier since there is only one isometry class in each dimension. In general, we see that the answer only depends of the Brauer class of A.

Apart from the split case, there are other reasons to believe that question 3 may have a positive answer. First, corollary 4.45 settles the case where i or j is equal to r in (30), and shows that we can take $N^r(r,j,d) = \delta_{d,r-j}$. We can also give an elementary treatment of the case $i = j = 1$.

Proposition 4.55. *Let (A,σ) be an algebra with involution over K, and let $(V,h) \in SW_\varepsilon(A,\sigma)$ of reduced dimension $r \in \mathbb{N}^*$. Then in $\widetilde{GW}(A,\sigma)$ we have*

$$h^2 = \left(\varepsilon T_B - \binom{r}{2}\mathcal{H}\right) + 2\lambda^2(h)$$

where T_B is the trace form of the algebra $B = \mathrm{End}_A(V)$.

Proof. The fundamental observation is that if $\tau = \sigma_h$, $T_\tau = T_\tau^+ + T_\tau^-$ while $T_B = T_\tau^+ + \langle -1\rangle T_\tau^-$. The formula follows from basic manipulation after injecting the relations $h^2 = T_\tau$ and $\lambda^2(h) = \langle 2\rangle T_\tau^{-\varepsilon}$. □

5 Mixed cohomology

In this section, we define a fundamental filtration (actually two) of $\widetilde{W}(A,\sigma)$, analogous to the fundamental filtration of $W(K)$, and study some of the basic properties of its associated graded ring, which we see as an analogue of the cohomology ring $H^*(K,\mathbb{Z}/2\mathbb{Z})$ (which we denote simply $H^*(K)$).

5.1 The fundamental filtration

Recall from proposition 2.32 and (2.8) the ring morphisms

$$\widetilde{\mathrm{rdim}_2} : \widetilde{W}(A,\sigma) \longrightarrow \mathbb{Z}/2\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$$

and

$$\mathrm{rdim}_2 : \widetilde{W}(A,\sigma) \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

Definition 5.1. *Let (A,σ) be an algebra with involution over K. Then we call*

$$I(A,\sigma) = \mathrm{Ker}(\mathrm{rdim}_2)$$
$$\widetilde{I}(A,\sigma) = \mathrm{Ker}(\widetilde{\mathrm{rdim}_2})$$

respectively the fundamental ideal and the homogeneous fundamental ideal of $\widetilde{W}(A,\sigma)$. The filtrations $(I^n(A,\sigma))_n$ and $(\widetilde{I}^n(A,\sigma))_n$ are respectively called the fundamental filtration and the homogeneous fundamental filtration.

By construction, $I(A,\sigma)$ is a maximal ideal such that $I(A,\sigma) \cap W(K) = I(K)$, and $\widetilde{I}^n(A,\sigma)$ is a homogeneous ideal. Note that when A is not split, then since its index is even we see that by definition $\mathrm{rdim}_2 = \widetilde{\mathrm{rdim}_2}$, so the homogeneous and non-homogenous filtrations actually coincide. On the other hand, when A is split the situation is quite different, and it is not clear which version to use. The homogeneous one allows a uniform treatment of split and non-split algebras, but the non-homogeneous one is arguably more natural, since $I(A,\sigma)$ is the unique prime ideal of $\widetilde{W}(A,\sigma)$ of residual characteristic 2 (see proposition 6.18). For those reasons, we will discuss both versions here.

We start with the homogeneous version. For any $n \in \mathbb{N}$, we write

$$\widetilde{I}^n(A,\sigma) = I_0^n(A,\sigma) \oplus I_+^n(A,\sigma) \oplus I_-^n(A,\sigma)$$

for the homogeneous decompositions of those ideals with respect to Γ. Note that of course $I_0^0(A,\sigma) = W(K)$ and $I_\varepsilon^0(A,\sigma) = W_\varepsilon(A,\sigma)$. Furthermore, $I_0(A,\sigma) = I(K)$, and if A is not split then $I_\varepsilon(A,\sigma) = W_\varepsilon(A,\sigma)$, while on the other hand $I_+(K,\mathrm{Id}) = I(K)$ and $I_-(K,\mathrm{Id}) = 0$.

Proposition 5.2. *Let (A,σ) be an algebra with involution over K, and let $n \in \mathbb{N}^*$. Then $\widetilde{I}^n(A,\sigma) = I^{n-1}(K)\widetilde{I}(A,\sigma)$.*

Proof. The formulas for $n = 1$ follow directly from the definition, as we noted just before the proposition. The proof for a general $n \in \mathbb{N}^*$ can be easily obtained by induction of we can show the case $n = 2$. A quick examination shows that the only non-trivial part is that $I_\varepsilon(A,\sigma)I_\varepsilon(A,\sigma) \subset I^2(K)$. Let $x, y \in I_\varepsilon(A,\sigma)$. Then $xy \in I^2(K)$ iff $\mathrm{disc}(xy) = 1$. Now this may be checked after generic splitting of A, since if L is the function field of the Severi-Brauer variety of A, then the map $K^*/(K^*)^2 \to L^*/(L^*)^2$ is injective. But indeed, in the split case xy is a product of elements with even dimension, so its discriminant is trivial. $\square$

For the non-homogeneous version, it is enough to treat the split case, since when A is not split it coincides with the homogeneous one, and we naturally start with the case $(A,\sigma) = (K, \mathrm{Id})$. Then we can use a general lemma:

Lemma 5.3. *Let R be a commutative ring, and let $J \subset R$ be an ideal. We define two ideals in $R[\mathbb{Z}/2\mathbb{Z}]$: $\widetilde{I} = J \oplus J$ and $I = \varepsilon^{-1}(J)$ (where ε is the augmentation map). Then for any $n \in \mathbb{N}^*$, we have $\widetilde{I}^n \subset I^n \subset \widetilde{I}^{n-1}$, and*

$$\begin{aligned}\widetilde{I}^n &= J^n \oplus J^n \\ I^n &= \widetilde{I}^{n-1} \cap \varepsilon^{-1}(J^n).\end{aligned}$$

Proof. The formula for $\widetilde{I}^n$ is easily proved by induction on n, and $\widetilde{I}^n \subset I^n$ is obvious since $\widetilde{I} \subset I$. We show the formula for I^n by induction. The case $n = 1$ is the definition. Suppose the formula holds up to $n \in \mathbb{N}^*$.

Let $x \in I^n$ and $y \in I$. Then $x = (x_0, x_1)$ with $x_0, x_1 \in J^{n-1}$ and $x_0+x_1 \in J^n$ by induction hypothesis, and $y = (y_0, y_1)$ with $y_0 + y_1 \in J$. Therefore, a direct computation shows that $xy = (z_0, z_1)$ with $z_0, z_1 \in J^n$ and $z_0 + z_1 \in J^{n+1}$.

Conversely, let $x \in \widetilde{I}^n \cap \varepsilon^{-1}(J^{n+1})$, and write $x = (x_0, x_1)$. Then we can write $x_0 = ay_0$ with $a \in J$ and $y_0 \in J^{n-1}$, and $x_0 + x_1 = by_1$ with $b \in J$ and $y_1 \in J^n$. Define $y = (y_0, -y_0)$ and $y' = (0, y_1)$; by induction hypothesis we have $y, z \in I^n$. We can then conclude since $x = ay + by'$. $\square$

We can then apply that to $\widetilde{W}(K, \mathrm{Id}) \simeq W(K)[\mathbb{Z}/2\mathbb{Z}]$. Let us write π_0 and π_1 for the projections of $\widetilde{W}(K, \mathrm{Id})$ on respectively the even and odd component. In particular, the augmentation map is $\varepsilon = \pi_0 + \pi_1$.

Proposition 5.4. *Let $n \in \mathbb{N}^*$. Then*

$$I^n(K, \mathrm{Id}) = \{(q_0, q_1) \in I^{n-1}(K) \oplus I^{n-1}(K) \mid q_0 + q_1 \in I^n(K)\}.$$

In particular, the filtration $(I^d(K, \mathrm{Id}))_d$ is separated and there is a natural isomorphism

$$I^n(K, \mathrm{Id}) \xrightarrow{(\varepsilon,\pi_0)} I^n(K) \oplus I^{n-1}(K).$$

Proof. The statement about $I^n(K, \mathrm{Id})$ is a direct application of lemma 5.3, with $R = W(K)$ and $J = I(K)$. It is then immediate that (ε, π_0) is a group isomorphism from $I^n(K, \mathrm{Id})$ to $I^n(K) \oplus I^{n-1}(K)$. The statement about the separation of the filtration then follows from the Hauptsatz. $\square$

5.2 Homogeneous mixed cohomology

Definition 5.5. *Let (A, σ) be an algebra with involution over K. We define the homogeneous mixed cohomology ring $\widetilde{H}^*(A, \sigma)$ as the graded ring associated to the homogeneous fundamental filtration. The group $\widetilde{H}^n(A, \sigma)$ is called the nth homogeneous mixed cohomology group of (A, σ), and we write*

$$\widetilde{e}_n : \widetilde{I}^n(A, \sigma) \longrightarrow \widetilde{H}^n(A, \sigma)$$

for the canonical projection.

Then $\widetilde{H}^*(A,\sigma)$ is a commutative $(\Gamma \times \mathbb{Z})$-graded $H^*(K)$-algebra, which is functorial in (A,σ) with respect to $\mathbf{Br}_h(K)$. We write the homogeneous decomposition:

$$\widetilde{H}^n(A,\sigma) = H_0^n(A,\sigma) \oplus H_+^n(A,\sigma) \oplus H_+^n(A,\sigma).$$

We see that $\widetilde{H}^0(A,\sigma)$ is $\mathbb{Z}/2\mathbb{Z}$ if A is not split, and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if A is split.

Proposition 5.6. *Let (A,σ) be an algebra with involution over K. Then*

$$\widetilde{H}^*(A,\sigma) = \widetilde{H}^0(A,\sigma) \oplus H^*(K)\widetilde{H}^1(A,\sigma).$$

In particular, $H_0^n(A,\sigma) = H^n(K)$ for any $n \in \mathbb{N}^*$.

Proof. This is a direct consequence of proposition 5.2. □

Proposition 5.7. *Let (A,σ) be an algebra with involution over K. Then the standard automorphisms of $\widetilde{W}(A,\sigma)$ induce the identity on $\widetilde{H}^*(A,\sigma)$. In particular, $\widetilde{H}^*(A,\sigma)$ only depends on the Brauer class of A, up to a canonical isomorphism.*

Proof. A standard automorphism acts by multiplication by some $\langle\lambda\rangle$ on $I_\varepsilon^n(A\sigma)$, but clearly $\langle\lambda\rangle x \equiv x$ modulo $I_\varepsilon^{n+1}(A,\sigma)$. □

If we wish to perform computations in $\widetilde{H}^*(A,\sigma)$, it is natural to introduce the equivalents of the Galois symbols.

Definition 5.8. *Let (A,σ) be an algebra with involution over K of even degree. For any $a \in A^*$ ε-symmetric, and $a_1,\dots,a_n \in K^*$, we set*

$$\langle\langle a_1,\dots,a_n;a\rangle\rangle_\sigma = \langle\langle a_1,\dots,a_n\rangle\rangle \cdot \langle a\rangle_\sigma \in \widetilde{I}^{n+1}(A,\sigma)$$

and we call such an element a mixed Pfister form over (A,σ). We also define

$$(a_1,\dots,a_n;a)_\sigma = \widetilde{e_{n+1}}(\langle\langle a_1,\dots,a_n;a\rangle\rangle_\sigma) = (a_1,\dots,a_n) \cup (a)_\sigma \in \widetilde{H}^{n+1}(A,\sigma)$$

and call such elements mixed symbols.

We call the usual Galois symbols in $H^*(K)$ *pure symbols*, and we use the generic term "symbol" for both pure and mixed symbols in $\widetilde{H}^*(A,\sigma)$.

Remark 5.9. If A is a division algebra then the symbols additively generate $\widetilde{H}^*(A,\sigma)$, but it is not the case in general.

Now to get a good understanding of $\widetilde{H}^n(A,\sigma)$, at least when A is a division algebra (which is enough by functoriality), we would need to find the relations between the symbols, analogously to what Milnor's conjecture does for pure symbols. This is still a work in progress. We can however give a good description of the cup-products of symbols. We clearly have

$$(a_1,\dots,a_n;a)_\sigma \cup (b_1,\dots,b_m;b)_\sigma = (a_1,\dots,b_m) \cup ((a)_\sigma \cup (b)_\sigma)$$

so it is enough to compute the cup-products of 1-symbols $(a)_\sigma \in \widetilde{H}^1(A,\sigma)$.

Proposition 5.10. *Let (A, σ) be an algebra with involution of degree $n = 2m$ over K. Let $\langle a\rangle_\sigma, \langle b\rangle_\sigma \in W_\varepsilon(A, \sigma)$. Then*

$$(a)_\sigma \cup (b)_\sigma = \begin{cases} (-\det(\sigma)\operatorname{Nrd}_A(a), -\operatorname{Nrd}_A(ab)) + m[A] \in H^2(K) & \text{if } \varepsilon = 1 \\ m[A] \in H^2(K) & \text{if } \varepsilon = -1. \end{cases}$$

Proof. The class in $H^2(K)$ we need to compute is exactly the Clifford invariant of $\langle a\rangle_\sigma \langle b\rangle_\sigma = T_{\sigma,a,b}$. Since its dimension is divisible by 4 and its discriminant is trivial, this is the same as its Hasse invariant (which is the second Stiefel-Whitney class). Let τ be the adjoint involution on $M_2(A)$ of the form $\langle a, b\rangle_\sigma$. Then the first step is recognizing that

$$T_\tau^+ = T_{\sigma_a}^+ + \langle 2\rangle T_{\sigma,a,b} + T_{\sigma_b}^+.$$

This is just a rephrasing of

$$\lambda^2(\langle a, b\rangle_\sigma) = \lambda^2(\langle a\rangle_\sigma) + \langle a\rangle_\sigma \langle b\rangle_\sigma + \lambda^2(\langle b\rangle_\sigma)$$

using propositions 2.28 and 4.33. Since the Clifford invariant only depends on the Witt class we get:

$$\begin{aligned} e_2(T_{\sigma,a,b}) &= w_2(\langle -1\rangle T_\tau^+ + T_{\sigma_a}^+ + T_{\sigma_b}^+) \\ &= w_2(\langle -1\rangle T_\tau^+) + w_2(T_{\sigma_a}^+) + w_2(T_{\sigma_b}^+) \\ &\quad + (\det(T_\tau^+), \det(T_{\sigma_a}^+)) + (\det(T_\tau^+), \det(T_{\sigma_b}^+)) + (\det(T_{\sigma_a}^+), \det(T_{\sigma_b}^+)). \end{aligned}$$

Now the Hasse invariants and the determinants of forms of type T_θ^+ have been computed by Quéguiner, see [18, §2.2,2.3]. We can collect the results:

$$\begin{aligned} \det(T_{\sigma_a}^+) &= 2^m \operatorname{Nrd}_A(a)\det(\sigma) \\ \det(T_\tau^+) &= \operatorname{Nrd}_A(ab) \\ w_2(T_{\sigma_a}^+) &= \begin{cases} \binom{m}{2}(-1,-1) + \binom{m}{2}[A] & \text{if } \varepsilon = -1 \\ (m+1)(-2, \det(\sigma)\operatorname{Nrd}_A(a)) + \binom{m+1}{2}[A] & \text{if } \varepsilon = 1 \end{cases} \\ w_2(\langle -1\rangle T_\tau^+) &= \begin{cases} m[A] & \text{if } \varepsilon = -1 \\ (2, \operatorname{Nrd}_A(ab)) + m(-1,-1) + m[A] & \text{if } \varepsilon = 1. \end{cases} \end{aligned}$$

The result then follows by a direct computation, using the fact that $(-1, 2) = 0$ and that if $\varepsilon = -1$ then $\operatorname{Nrd}_A(a)$ and $\operatorname{Nrd}_A(b)$ have the same square class, which coincides with $\det(\sigma)$ (see [11, 7.1]). □

Remark 5.11. Note that the formula is indeed symmetric in a and b, since $(-\det(\sigma)\operatorname{Nrd}_A(a), -\operatorname{Nrd}_A(ab)) = (-\det(\sigma)\operatorname{Nrd}_A(b), -\operatorname{Nrd}_A(ab))$.

Example 5.12. If A is not split, then there is a symplectic involution θ on the division algebra D Brauer-equivalent to A, and we can reduce to computations in $\widetilde{H}^*(A, \sigma)$ to computations in $\widetilde{H}^*(D, \theta)$, and according to proposition 5.7 the isomorphism does not depend on the choice of Morita equivalence. Then if D is a quaternion algebra, $(a)_\theta \cup (b)_\theta$ is $[D]$ if $a, b \in K^*$, and $(a^2, b^2) + [D]$ if a and b are pure quaternions. This is coherent with proposition 3.22.

If D is not a quaternion algebra, then $(a)_\theta \cup (b)_\theta$ is 0 if a and b are symmetric, and $(-Nrd_D(a), -\operatorname{Nrd}_D(b))$ if a and b are anti-symmetric.

Corollary 5.13. *Let (A, σ) be an algebra with involution of even degree over K. Then a product of symbols in $\widetilde{H}^*(A, \sigma)$ is a symbol, except possibly if A is a non-division algebra of index 2 and odd coindex.*

Proof. We only have to check that $(a)\sigma \cup (b)_\sigma$ is a Galois symbol. If A is split, then this follows immediately from proposition 5.10. If the index of A is strictly greater than 2, then its degree is divisible by 4, so it also follows from the proposition. Now assume the index is 2. Then the m in the proposition is the coindex, so if it is even then we still get a symbol. Finally, if A is a division algebra, so $m = 1$ and A is a quaternion algebra, then the result follows from the observation that if z_1 and z_2 are non-zero pure quaternions, (z_1^2, z_2^2) and $[A]$ have a common slot (for instance z_1^2), so their sum is a symbol (or directly from proposition 3.22). □

5.3 Non-homogeneous mixed cohomology

We make similar definitions as in the homogeneous case.

Definition 5.14. *Let (A, σ) be an algebra with involution over K. We define the mixed cohomology ring $H^*(A, \sigma)$ as the graded ring associated to the fundamental filtration. The group $H^n(A, \sigma)$ is called the nth mixed cohomology group of (A, σ), and we write*

$$e_n : I^n(A, \sigma) \longrightarrow H^n(A, \sigma)$$

for the canonical projection.

Then $H^*(A, \sigma)$ is a commutative $\mathbb{Z}$-graded $H^*(K)$-algebra, which is functorial in (A, σ) with respect to $\mathbf{Br}_h(K)$, so in particular it only depends on the Brauer class of A up to a *non-canonical* isomorphism (this is different from the homogeneous case). We see that $H^0(A, \sigma)$ is $\mathbb{Z}/2\mathbb{Z}$ whether or not A is split. Of course if A is not split $H^n(A, \sigma) = \widetilde{H}^n(A, \sigma)$ and all the previous results apply. Thus we will focus more on the split case.

Proposition 5.15. *Let $n \in \mathbb{N}^*$. There is a commutative diagram where the horizontal maps are isomorphisms:*

$$\begin{array}{ccc} I^n(K, \mathrm{Id}) & \xrightarrow{(\varepsilon, \pi_0)} & I^n(K) \oplus I^{n-1}(K) \\ {\scriptstyle e_n}\downarrow & & \downarrow{\scriptstyle e_n \oplus e_{n-1}} \\ H^n(K, \mathrm{Id}) & \xrightarrow{(\partial_1, \partial_2)} & H^n(K) \oplus H^{n-1}(K). \end{array}$$

Proof. The first line has been established in proposition 5.4. The second line is then a direct consequence using $H^n(K) = I^n(K)/I^{n+1}(K)$. The vertical maps make the diagram commute by definition. □

Remark 5.16. In the above diagram we could have used π_1 instead of π_0. Indeed, we can characterize $I^n(K, \mathrm{Id})$ be the fact that $e_{n-1}(\pi_0(x))$ and $e_{n-1}(\pi_1(x))$ exist and are equal, and then $\partial_2(e_n(x))$ is this common value.

Corollary 5.17. *Let (A, σ) be a split algebra with involution over K. Then the fundamental filtration of $\widetilde{W}(A, \sigma)$ is separated. Furthermore, any choice* *of isomorphism* $f : (A,\sigma) \to (K, \mathrm{Id})$ *in* $\mathbf{Br}_h(K)$ *induces an isomorphism of* $\mathbb{Z}$*-graded* $H^*(K, \mathbb{Z}/2\mathbb{Z})$*-modules:*

$$H^*(A,\sigma) \xrightarrow{(\partial_f,\partial_2)} H^*(K) \oplus H^*(K)[-1].$$

The morphism ∂_2 *is actually independent of the choice of* f*, while* $\partial_f(x)$ *is only well-defined modulo* $\partial_2(x) \cup H^1(K, \mathbb{Z}/2\mathbb{Z})$*. Precisely:*

$$\partial_{\langle\lambda\rangle f}(x) = \partial_f(x) + (\lambda) \cup \partial_2(x).$$

Proof. Since $I(A,\sigma)$ is functorial in (A,σ) (relative to $\mathbf{Br}_h(K)$), the fact that the filtration is separated follows from the case of (K, Id). Similarly the existence of the isomorphism is a direct consequence of proposition 5.4 since we are choosing an isomorphism $f_* : \widetilde{W}(A,\sigma) \xrightarrow{\sim} \widetilde{W}(K, \mathrm{Id})$.

Taking $f' = \langle\lambda\rangle f$ amounts to modifying $\widetilde{W}(K, \mathrm{Id})$ by the standard automorphism $\langle\lambda\rangle_*$, which means multiplying the odd component by $\langle\lambda\rangle$. We can then conclude if we consider the following commutative diagram

$$\begin{array}{ccc} I^n(K,\mathrm{Id}) & \xrightarrow{\langle\lambda\rangle_*} & I^n(K,\mathrm{Id}) \\ {\scriptstyle(\varepsilon,\pi_0)}\downarrow & & \downarrow{\scriptstyle(\varepsilon,\pi_0)} \\ I^n(K) \oplus I^{n-1}(K) & \xrightarrow{\alpha} & I^n(K) \oplus I^{n-1}(K) \end{array}$$

with $\alpha(x,y) = (\langle\lambda\rangle x + \langle\langle\lambda\rangle\rangle y, y)$, which is easy to establish. □

Remark 5.18. In particular, if we write $H^*_h(A,\sigma) = \mathrm{Ker}(\partial_2)$, then we have a well-defined isomorphism $\partial_1 : H^*_h(A,\sigma) \xrightarrow{\sim} H^*(K)$ (the h stands for "homogeneous", see proposition 5.19) for an explanation).

We now give a comparison of homogeneous and non-homogeneous cohomology, especially in the split case (since otherwise they coincide). From lemma 5.3, we see that in the split case

$$\widetilde{I}^n(A,\sigma) \subset I^n(A,\sigma) \subset \widetilde{I}^{n-1}(A,\sigma),$$

and in the non-split case this is even more obvious since the first inclusion is an equality. In particular there are natural maps

$$\widetilde{H}^n(A,\sigma) \longrightarrow H^n(A,\sigma) \longrightarrow \widetilde{H}^{n-1}(A,\sigma).$$

When A is not split, they are, from left to right, the identity and the zero map.

Proposition 5.19. *Let* (A,σ) *be a split algebra with involution over* K*. Then the image of* $\widetilde{H}^*(A,\sigma) \to H^*(A,\sigma)$ *is* $H^*_h(A,\sigma)$*. More precisely, for any choice of* $f : (A,\sigma) \to (K, \mathrm{Id})$ *in* $\mathbf{Br}_h(K)$*, we have a commutative diagram of* $\mathbb{Z}$*-graded* $H^*(K)$*-modules, where the vertical maps are isomorphisms:*

$$\begin{array}{ccccc} \widetilde{H}^*(A,\sigma) & \longrightarrow & H^*(A,\sigma) & \longrightarrow & \widetilde{H}^*(A,\sigma)[-1] \\ \downarrow{\scriptstyle(\pi_0,\pi_1)} & & \downarrow{\scriptstyle(\partial_f,\partial_2)} & & \downarrow{\scriptstyle(\pi_0,\pi_1)} \\ H^*(K) \oplus H^*(K) & \xrightarrow{\alpha} & H^*(K) \oplus H^*(K)[-1] & \xrightarrow{\beta} & H^*(K)[-1] \oplus H^*(K)[-1] \end{array}$$

with $\alpha(x,y) = (x+y, 0)$ *and* $\beta(x,y) = (y,y)$*.*

Proof. Note that for the degree 0 components, our description of $H^0(A,\sigma)$ and $\widetilde{H}^0(A,\sigma)$ shows that the first row is

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

which corresponds to the second row. We now look at the degree n components, for any $n \in \mathbb{N}^*$.

We can use functoriality through f to reduce to the diagram

$$\begin{array}{ccccc} \widetilde{H}^n(K,\mathrm{Id}) & \longrightarrow & H^n(K,\mathrm{Id}) & \longrightarrow & \widetilde{H}^{n-1}(K,\mathrm{Id}) \\ \downarrow{\scriptstyle(\pi_0,\pi_1)} & & \downarrow{\scriptstyle(\partial_1,\partial_2)} & & \downarrow{\scriptstyle(\pi_0,\pi_1)} \\ H^n(K)\oplus H^n(K) & \xrightarrow{\alpha} & H^n(K)\oplus H^{n-1}(K) & \xrightarrow{\beta} & H^{n-1}(K)\oplus H^{n-1}(K). \end{array}$$

To show that the left-hand square commutes, take $x = \widetilde{e_n}(q_0,q_1) \in \widetilde{H}^n(K,\mathrm{Id})$ (so $q_0, q_1 \in I^n(K)$). Then by definition of ∂_1 and ∂_2, if we go through the bottom path we get $(e_n(q_0)+e_n(q_1),0)$, and if we go through the upper path we get $(e_n(q_0+q_1), e_{n-1}(q_0))$, so indeed the square commutes.

For the right-hand square, take $x = e_n(q_0,q_1) \in H^n(K,\mathrm{Id})$ (so $q_0,q_1 \in I^{n-1}(K)$ and $q_0+q_1 \in I^n(K)$). The going through the bottom path we get $(e_{n-1}(q_0), e_{n-1}(q_0))$, and going through the upper path we get $(e_{n-1}(q_0), e_{n-1}(q_1))$, so this square commutes too. □

Corollary 5.20. *Let (A,σ) be an algebra with involution over K. Then there is a natural exact complex of $\mathbb{Z}$-graded $H^*(K)$-modules*

$$\cdots \to \widetilde{H}^*(A,\sigma)[n+1] \to H^*(A,\sigma)[n+1] \to \widetilde{H}^*(A,\sigma)[n] \to H^*(A,\sigma)[n] \to \dots$$

Proof. The only thing to prove is the exactness. If A is not split then every other map is the identity and the other ones are zero. If A is split, then this follows directly from proposition 5.19, since the exactness is immediate on the bottom row (even if we extend the sequence to an arbitrary number of terms). □

6 Spectrum and signatures

The theory of orderings of a field, as initiated by Artin and Schreier in [1], has strong ties with the structure of the Witt ring of the field, through the study of signatures. Various efforts have been made to extend this theory of signatures to involutions (in [14] for involutions of the first kind, and [17] for involutions of the second kind, see also [11, §11]) and hermitian forms, most notably by Astier and Unger in a series of articles (mainly [2] and [3]).

The goal of this section is to expose how this theory fits in the framework of mixed Witt rings, imitating the classical Artin-Schreier theory. While we do not prove many fundamentally new result about signatures of hermitian forms, we do provide a new point of view on a number of previously known results and constructions, and we argue that this point of view at the very least sheds an interesting light on the results of Astier and Unger, and allows for more satisfying statements in some cases (however, we only treat the case of involutions of the first kind). As an example of improvement, previously much of the focus has been on the definition of an appropriate total signature of an hermitian form, which would be a (preferably continuous) function from the set of orderings $X(K)$ to $\mathbb{Z}$, but this has always necessitated some arbitrary choices; we define a canonical total signature $\widetilde{\mathrm{sign}}(x)$ which is a continuous function from $\widetilde{X}(A,\sigma)$ to $\mathbb{Z}$, where $\widetilde{X}(A,\sigma)$ is a canonical double-cover of $X(K)$, and only then we investigate how to make pertinent choices to obtain a (non-canonical) total signature defined on $X(K)$.

6.1 Orderings and signatures over fields

We start with a brief overview of the theory over fields, and we refer to [12] for proofs.

Definition 6.1. *A field K is formally real if -1 is not a sum of squares in K; it is real closed if in addition no algebraic extension of K is formally real.*

An ordering on a field K is a subgroup $P \subset K^*$ of index 2 which is stable under addition and does not contain -1. We say that (K,P) is an ordered field, we write $\mathrm{sign}_P : K^* \to \{\pm 1\}$ the morphism with kernel P, and $\mathrm{sign}_P(a)$ is called the P-sign (or the sign if no confusion is possible) of $a \in K^*$. We can then speak of P-positive and P-negative elements.

We write $X(K)$ for the set of all orderings of K.

An extension of an ordered field (K,P) is an ordered field (L,Q) such that L/K is an extension with $P = Q \cap K$. If L/K is algebraic and (L,Q) is real closed, then (L,Q) is called a real closure of K.

Proposition 6.2. *A field K admits an ordering iff it is formally real; a real closed field admits a unique ordering. Any ordered field (K,P) admits a real closure K_P, unique up to a unique K-isomorphism.*

Proposition 6.3. *If L is real closed, then there is a (unique) ring isomorphism between $W(L)$ and $\mathbb{Z}$, sending $\langle a\rangle$ to its sign relative to the unique ordering of L, for all $a \in L^*$.*

Thus for any ordering P on a field K, there is a unique ring morphism

$$\mathrm{sign}_P : W(K) \longrightarrow W(K_P) \xrightarrow{\sim} \mathbb{Z},$$

called the *signature* of K at P, which extends the P-sign map on K^* (meaning that $\mathrm{sign}_P(\langle a\rangle) = \mathrm{sign}_P(a)$). For any $p \in \mathbb{N}$ that is either 0 or a prime number, we write

$$\mathrm{sign}_{P,p} : W(K) \xrightarrow{\mathrm{sign}_P} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}.$$

Then we set

$$I_{P,p}(K) = \mathrm{Ker}(\mathrm{sign}_{P,p}).$$

We also write $I_P(K) = I_{P,0}(K)$.

Proposition 6.4. *For any ordering P on K, we have $I_{P,2}(K) = I(K)$. Furthermore,*

$$\mathrm{Spec}(W(K)) = \{I(K)\} \coprod_{P \in X(K)} \{I_{P,p}(K) \,|\, p \text{ odd or } 0\}.$$

In particular, there is a canononical identification between $X(K)$ and $\mathrm{Spec}_0(W(K))$.

Remark 6.5. If K is formally real, $X(K)$ is also identified with minSpec($W(K)$) (otherwise, $X(K)$ is empty and $\mathrm{minSpec}(W(K)) = \{I(K)\}$). This endows $X(K)$ with its so-called *Harrison topology.* In particular, $X(K)$ is Hausdorff and totally disconnected. In addition, it can be shown that it is compact, since the embedding $X(K) \to \{\pm 1\}^{K^*}$ given by $P \mapsto \mathrm{sign}_P$ is actually a closed immersion.

6.2 Canonical retractions

The fundamental ingredient in the definition of signature maps on mixed Witt rings is the existence of certain natural ring morphisms:

Definition 6.6. *Let (A, σ) be an algebra with involution over K. We say that a ring morphism $\rho : \widetilde{W}(A, \sigma) \to W(K)$ is a retraction of $\widetilde{W}(A, \sigma)$ if it is the identity on $W(K)$. We define similarly retractions of $\widetilde{W}_\varepsilon(A, \sigma)$ which we also call orthogonal and symplectic retractions of $\widetilde{W}(A, \sigma)$ (depending on ε).*

Example 6.7. The augmentation map $W(K)[\mathbb{Z}/2\mathbb{Z}] \to W(K)$ defines a retraction ρ of $\widetilde{W}(K, \mathrm{Id})$, which we call the canonical retraction of $\widetilde{W}(K, \mathrm{Id})$.

We denote by $\mathbb{H}_K$ the Hamilton quaternion algebra over K, so that its Brauer class is $[\mathbb{H}_K] = (-1, -1) \in H^2(K, \mu_2)$. Recall that a classical result of Frobenius (at least for the real numbers) states that the only central division algebras over a real closed field are K and $\mathbb{H}_K$, which explains why the Hamilton quaternions play a crucial role in our exposition.

Proposition 6.8. *Let K be a real closed field. Then there is a (unique) retraction ρ of $\widetilde{W}(\mathbb{H}_K, \gamma)$, called the canonical retraction, such that $\rho(\langle 1\rangle_\gamma) = 2$ and $\rho(\langle z\rangle_\gamma) = 0$ for any non-zero pure quaternion $z \in \mathbb{H}_K$.*

Proof. The uniqueness of ρ is clear since $W^{\pm}(\mathbb{H}_K, \gamma)$ is generated as a $W(K)$-module by $\langle 1\rangle_\gamma$ and the $\langle z\rangle_\gamma$.

In general, for any quaternion algebra Q over any field K, if $(V, h) \in W(Q, \gamma)$, then there is a natural quadratic form $q_h : V \to K$ defined by $q_h(x) = h(x, x)$, and it is easy to see that if $h = \langle a_1, \dots, a_n\rangle_\gamma$ then $q_h = \langle a_1, \dots, a_n\rangle n_Q$. In our case, since $n_{\mathbb{H}_K} = \langle\langle -1, -1\rangle\rangle$, this means that $\langle a_1, \dots, a_n\rangle_\gamma \mapsto 4\langle a_1, \dots, a_n\rangle$ is a well defined $W(K)$-module morphism $W(\mathbb{H}_K, \gamma) \to W(K)$. Since $W(K)$ is torsion-free, we can divide by 2 and get a morphism sending $\langle 1\rangle_\gamma$ to 2. Thus there is a unique $W(K)$-module morphism ρ satisfying the conditions of the statement.

Clearly $\rho(xy) = \rho(x)\rho(y) = 0$ if $x \in W(\mathbb{H}_K, \gamma)$ and $y \in W^-(\mathbb{H}_K, \gamma)$. We just need to check that $\rho(\langle a\rangle_\gamma \cdot \langle b\rangle_\gamma) = \rho(\langle a\rangle_\gamma)\rho(\langle b\rangle_\gamma)$ for all $a, b \in K^*$, and $\rho(\langle z_1\rangle_\gamma \cdot \langle z_2\rangle_\gamma) = \rho(\langle z_1\rangle_\gamma)\rho(\langle z_2\rangle_\gamma)$ for all $z_1, z_2 \in \mathbb{H}_K$ non-zero pure quaternions. According to proposition 3.22, this respectively means that

$$\langle 2ab\rangle n_{\mathbb{H}_K} = (2\langle a\rangle) \cdot (2\langle b\rangle)$$

and

$$\langle -\mathrm{Trd}_Q(z_1 z_2)\rangle \varphi_{z_1, z_2} = 0$$

(unless z_1 and z_2 anti-commute, in which case the condition is trivial). The first one is true because $n_{\mathbb{H}_K} = 4 \in W(K)$ and $\langle 2\rangle$ is represented by $n_{\mathbb{H}_K}$; the second one is true because φ_{z_1,z_2} is hyperbolic. Indeed, the square of any pure quaternion in $\mathbb{H}_K$ is negative, so $(z_1^2, z_2^2) = (-1,-1) = [\mathbb{H}_K]$, which by definition of φ_{z_1,z_2} means it is hyperbolic. $\square$

Remark 6.9. The proof shows that we can define a canonical retraction of $\widetilde{W}_{-1}(\mathbb{H}_K,\gamma)$ for any field K. Furthermore, we can define ρ on $\widetilde{W}(\mathbb{H}_K,\gamma)$ assuming only that K is Pythagorean. On the other hand, no retraction can exist on $\widetilde{W}(\mathbb{H}_K,\gamma)$ if the Pythagoras number of K is at least 3. We do not know whether there is always a retraction if the Pythagoras number is 2.

6.3 Signature maps

Since every central simple algebra over a real closed field is either split or Brauer-equivalent to the Hamilton quaternions, we can make the following definition:

Definition 6.10. *Let A be a central simple algebra over K. Then for any ordering $P \in X(K)$, we say that P is orthogonal with respect to A if A_{K_P} is split; otherwise, A_{K_P} is Brauer-equivalent to $\mathbb{H}_{K_P}$ and we say that P is symplectic with respect to A.*

The set of orthogonal orderings of K with respect to A is denoted $X_+(A)$, and the set of symplectic orderings is $X_-(A)$.

If $P \in X_+(A)$, we define $(D_P,\theta_P) = (K_P, \mathrm{Id})$; if $P \in X_-(A)$, then $(D_P,\theta_P) = (\mathbb{H}_{K_P},\gamma)$.

Example 6.11. If A is split, $X_+(A) = X(K)$ and $X_-(A) = \emptyset$. On the other hand, $X_+(\mathbb{H}_K) = \emptyset$ and $X_-(\mathbb{H}_K) = X(K)$.

Remark 6.12. In the terminology of [2], $Nil[A,\sigma]$ is $X_+(A)$ if σ is symplectic, and $X_-(A)$ if σ is orthogonal. It is shown in [2, cor 6.5] that $X_+(A)$ and $X_-(A)$ are clopen in $X(K)$, and in particular are compact and totally disconnected (we will also provide a proof).

Let (A,σ) be an algebra with involution over K. For any $P \in X(K)$, let us choose an arbitrary isomorphism $f_P : (A_{K_P},\sigma_{K_P}) \to (D_P,\theta_P)$ in $\mathbf{Br}_h(K_P)$. Then we define a ring morphism

$$\widetilde{\mathrm{sign}}_P^+ : \widetilde{W}(A,\sigma) \longrightarrow \widetilde{W}(A_{K_P},\sigma_{K_P}) \xrightarrow{(f_P)_*} \widetilde{W}(D_P,\theta_P) \xrightarrow{\rho} W(K_P) \xrightarrow{\mathrm{sign}} \mathbb{Z}$$

and call it a *signature map* of (A,σ) at P. Any other choice of f_P has the form $\langle a\rangle f_P$ for some $a \in K_P^*$. Since K_P is real closed, $\langle a\rangle = \pm\langle 1\rangle$ in $W(K_P)$, so there are only two possible choices for f_P, and the other choice (when a is negative) leads to a different map $\widetilde{\mathrm{sign}}_P^-$; they are both equal to sign_P on $W(K)$, and they differ by a sign on $W^\pm(A,\sigma)$. The set $\{\widetilde{\mathrm{sign}}_P^+, \widetilde{\mathrm{sign}}_P^-\}$ is well-defined, but the labels $+$ and $-$ depend on the choice of f_P and are a priori arbitrary (but see section 6.5). When a statement does not depend on the choice of labels we will sometimes write $\widetilde{\mathrm{sign}}_P^\pm$ to designate either of the two possible maps.

Note that by construction of the canonical retractions ρ, $\widetilde{\mathrm{sign}}_P^\pm$ are both zero on $W_+(A,\sigma)$ if $P \in X_-(A)$ and are zero on $W_-(A,\sigma)$ if $P \in X_+(A)$. On the other hand, Astier and Unger show:

Lemma 6.13 ([2], thm 6.1)**.** *Let $\varepsilon = \pm 1$ and let $P \in X_\varepsilon(A)$. Then there exists $h \in W_\varepsilon(A,\sigma)$ such that $\widetilde{\mathrm{sign}}_P^{\pm}(h) \neq 0$. In particular, $\widetilde{\mathrm{sign}}_P^{+}$ and $\widetilde{\mathrm{sign}}_P^{-}$ are different functions on $\widetilde{W}(A,\sigma)$.*

We can now check that our construction is exhaustive:

Proposition 6.14. *Let (A,σ) be an algebra with involution over K, and let $\varepsilon, \varepsilon' = \pm 1$. For any $P \in X_\varepsilon(K)$, the only ring morphisms $\widetilde{W}_{\varepsilon'}(A,\sigma) \to \mathbb{Z}$ that extend the signature* sign_P *on $W(K)$ are $\widetilde{sign}_P^{+}$ and $\widetilde{\mathrm{sign}}_P^{-}$ (with arbitrary labels $+$ and $-$).*

If $\varepsilon = \varepsilon'$ then these two morphisms are distinct on $\widetilde{W}_{\varepsilon'}(A,\sigma)$, while if $\varepsilon \neq \varepsilon'$ they coincide on $\widetilde{W}_{\varepsilon'}(A,\sigma)$ and are both zero on $W_{\varepsilon'}(A,\sigma)$.

Proof. Let $f : \widetilde{W}_{\varepsilon'}(A,\sigma) \to \mathbb{Z}$ be a ring morphism extending sign_P, and let $x \in W_{\varepsilon'}(A,\sigma)$. Then $f(x)^2 = \mathrm{sign}_P(x^2) = \widetilde{\mathrm{sign}}_P^{\pm}(x)^2$, so $f(x) = \widetilde{\mathrm{sign}}_P^{s(x)}(x)$ for some $s(x) = \pm 1$. Now we want to show that we can take $s(x)$ constant.

If $\varepsilon \neq \varepsilon'$, we already noticed that both $\widetilde{\mathrm{sign}}_P^{\pm}$ are zero on $W_{\varepsilon'}(A,\sigma)$ so we can choose $s(x)$ arbitrarily for all x.

If $\varepsilon = \varepsilon'$, according to lemma 6.13, there is some $y \in W_{\varepsilon'}(A,\sigma)$ such that $\widetilde{\mathrm{sign}}_P^{\pm}(y) \neq 0$; in particular, $s = s(y)$ is uniquely determined. Now for an arbitrary x, if $\widetilde{\mathrm{sign}}_P^{\pm}(x) = 0$ then we can choose $s(x)$ arbitrarily so we take $s(x) = s$, and if $\widetilde{\mathrm{sign}}_P^{\pm}(x) \neq 0$, then

$$f(xy) = \mathrm{sign}_P(xy) = \widetilde{\mathrm{sign}}_P^{s}(xy) = s(x)s \cdot \widetilde{\mathrm{sign}}_P^{s(x)}(x)\widetilde{\mathrm{sign}}_P^{s}(y) = s(x)s \cdot f(x)f(y)$$

and since $f(x)f(y) \neq 0$, we have $s(x)s = 1$. ☐

Remark 6.15. In [3, prop 7.4], lacking a a ring structure, Astier and Unger show a slightly stronger result (with the cost of a more involved proof): $\widetilde{\mathrm{sign}}_P^{+}$ is the only $W(K)$-module morphisms extending sign_P, where we see $\mathbb{Z}$ as a $W(K)$-module through sign_P, up to multiplication by an arbitrary integer on $W^{\pm}(A,\sigma)$. So the only thing that compatibility with the product of hermitian forms adds to our statement is a normalization condition (our morphisms can only differ by a sign on $W^{\pm}(A,\sigma)$ and not an arbitrary integer). On that subject, it should be noted that they normalize the signature maps at symplectic orderings so that they give surjective maps $W_-(A,\sigma) \to \mathbb{Z}$, while with our construction we get a map to $2\mathbb{Z}$ (which is necessary to get a ring morphism).

Corollary 6.16. *Let (A,σ) be an algebra with involution over K, and let $P \in X(K)$. There are exactly two different ring morphisms $\widetilde{W}(A,\sigma) \to \mathbb{Z}$ that extend the signature* sign_P *on $W(K)$, namely $\widetilde{sign}_P^{+}$ and $\widetilde{\mathrm{sign}}_P^{-}$ (with arbitrary labels $+$ and $-$).*

Remark 6.17. We can also define a signature of the involution σ: $\mathrm{sign}_P^{\pm}(\sigma) \overset{\mathrm{def}}{=} \widetilde{\mathrm{sign}}_P^{\pm}(\langle 1 \rangle_\sigma)$. We again encounter a sign ambiguity, which is why in [14] only the absolute value $|\mathrm{sign}_P^{\pm}(\sigma)|$ is defined, and taken as the definition of the signature of σ. Note that the definitions agree since they characterize $\mathrm{sign}_P(\sigma) \in \mathbb{N}$ by $\mathrm{sign}_P(\sigma)^2 = \mathrm{sign}_P(T_\sigma)$, and of course $T_\sigma = \langle 1 \rangle_\sigma^2$ in $\widetilde{W}(A,\sigma)$.

6.4 The spectrum of the mixed Witt ring

Now that we have our signature maps, we want to obtain a description of $\mathrm{Spec}(\widetilde{W}(A,\sigma))$ similar to proposition 6.4 for $W(K)$.

Let $P \in X(K)$ and let $p \in \mathbb{N}$ be either 0 or a prime number. Assume we chose a labelling of $\widetilde{\mathrm{sign}}_P^+$ and $\widetilde{\mathrm{sign}}_P^-$. Then we define

$$\widetilde{\mathrm{sign}}_{P,p}^{\pm} : \widetilde{W}(A,\sigma) \xrightarrow{\widetilde{\mathrm{sign}}_P^{\pm}} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$$

and

$$I_{P,p}^{\pm}(A,\sigma) = \mathrm{Ker}(\widetilde{\mathrm{sign}}_{P,p}^{\pm}),$$

which is by construction a prime ideal of $\widetilde{W}(A,\sigma)$ (maximal if $p \neq 0$).

Let $P \in X_\varepsilon(K)$. Then since $\widetilde{\mathrm{sign}}_P^{\pm}$ is zero on $W_{-\varepsilon}(A,\sigma)$, we can write

$$I_{P,p}^{\pm}(A,\sigma) = J_{P,p}^{\pm}(A,\sigma) \oplus W_{-\varepsilon}(A,\sigma)$$

with

$$J_{P,p}^{\pm}(A,\sigma) = I_{P,p}^{\pm}(A,\sigma) \cap W_\varepsilon(A,\sigma).$$

Proposition 6.18. *Let (A,σ) be an algebra with involution over K. Then $I(A,\sigma)$ is the only prime ideal of $\widetilde{W}(A,\sigma)$ with residual characteristic 2; in particular, for any $P \in X(K)$, we have $I_{P,2}^{\pm}(A,\sigma) = I(A,\sigma)$. Furthermore, consider the following natural commutative diagram of schemes:*

$$\begin{array}{ccccc} & & \mathrm{Spec}(\widetilde{W}(A,\sigma)) & & \\ & \swarrow & \downarrow \pi & \searrow & \\ \mathrm{Spec}(\widetilde{W}_+(A,\sigma)) & & & & \mathrm{Spec}(\widetilde{W}_-(A,\sigma)) \\ & \searrow^{\pi_+} & & \swarrow^{\pi_-} & \\ & & \mathrm{Spec}(W(K)). & & \end{array}$$

The fiber of π above $I(K) \in \mathrm{Spec}(W(K))$ is $\{I(A,\sigma)\}$.

Let $P \in X(K)$, and p be either 0 or an odd prime. The fiber of π above $I_{P,p}(K)$ is $\{I_{P,p}^+(A,\sigma), I_{P,p}^-(A,\sigma)\}$ (the two being distinct).

If $P \in X_\varepsilon(A)$, the fiber of π_ε above $I_{P,p}(K)$ is $\{J_{P,p}^+(A,\sigma), J_{P,p}^-(A,\sigma)\}$ (the two being distinct), while the fiber of $\pi_{-\varepsilon}$ is $\{I_{P,p}(K) \oplus W_{-\varepsilon}(A,\sigma)\}$.

Proof. Let $I \subset \widetilde{W}(A,\sigma)$ be a prime ideal with residual characteristic 2. Then $I \cap W(K)$ is a prime ideal with residual characteristic 2, so $I \cap W(K) = I(K)$. If $x \in W^{\pm}(A,\sigma)$, then $x \in I$ iff $x^2 \in I(K)$, which is equivalent to $\mathrm{rdim}_2(x)^2 = 0 \in \mathbb{Z}/2\mathbb{Z}$, so $x \in I(A,\sigma)$. So $I(A,\sigma) \subset I$, and since $I(A,\sigma)$ is a maximal ideal we have equality. This implies the statement about $I_{P,2}^{\pm}(A,\sigma)$ and about the fiber of π above $I(K)$.

Now let $P \in X_\varepsilon(K)$, and p be either 0 or an odd prime; we set $R = \mathbb{Z}/p\mathbb{Z}$. Let I be in the fiber of $\pi_{\varepsilon'}$ above $I_{P,p}(K)$, and let $f : \widetilde{W}_{\varepsilon'}(A,\sigma) \to S$ be the surjective morphism with kernel I, with $R \subset S$. Then the same proof as for proposition 6.14 shows that $R = S$ and $f = \widetilde{\mathrm{sign}}_{P,p}^{s}$ for some $s = \pm 1$ (which is uniquely determined when $\varepsilon = \varepsilon'$). Indeed, we show the same way that for fixed $x \in W_{\varepsilon'}(A,\sigma)$ we have $f(x)^2 = \widetilde{\mathrm{sign}}^{\pm}_{P,p}(x)^2 \in R$, which shows that $R = S$ since S is integral. The rest of the reasoning is also the same, the only difference being that we have to invoque that $p \neq 2$ to justify that we get two different signature maps when $\varepsilon = \varepsilon'$.

Now suppose I is in the fiber of π above $I_{P,p}(K)$. Then we just showed that $I \cap \widetilde{W}_{\varepsilon}(A,\sigma) = J^s_{P,p}$ for some $s = \pm 1$ and that $I \cap \widetilde{W}_{-\varepsilon}(A,\sigma) = I_{P,p}(K) \oplus W_{-\varepsilon}(A,\sigma)$. This shows that $I = I^s_{P,p}(A,\sigma)$. □

Remark 6.19. In the continuity of remark 6.15, Astier and Unger show in [3, 6.5,6.7] slightly different and arguably stronger results, since they obtain a similar classification without asking that their "ideals" be stable by multiplication by a hermitian form. There is however a difference for primes above $I(K)$, since they find many such "ideals" (but of course only $I(A,\sigma)$ is an actual ideal).

Emulating the classical case, we set

$$\widetilde{X}(A,\sigma) = \mathrm{Spec}_0(\widetilde{W}(A,\sigma))$$

as a topological subspace of $\mathrm{Spec}(\widetilde{W}(A,\sigma))$; its elements are the $I^{\pm}_P$ for $P \in X(K)$. When K is formally real, this is also $\mathrm{minSpec}(\widetilde{W}(A,\sigma))$ (otherwise, $\widetilde{X}(A,\sigma)$ is empty, while $\mathrm{minSpec}(\widetilde{W}(A,\sigma))$ is a single point). Thus the continuous map $\pi : \mathrm{Spec}(\widetilde{W}(A,\sigma)) \to \mathrm{Spec}(W(K))$ induces a continuous two-to-one map $\overline{\pi} : \widetilde{X}(A,\sigma) \to X(K)$. We also set $\widetilde{X}_{\varepsilon}(A,\sigma) = \pi^{-1}(X_{\varepsilon}(A))$, so that $\widetilde{X}_{\varepsilon}(A,\sigma) \to X_{\varepsilon}(A)$ is also a continuous two-to-one map. We easily see from proposition 6.18 that $\mathrm{Spec}_0(\widetilde{W}_{\varepsilon}(A,\sigma))$ is canonically identified with $\widetilde{X}_{\varepsilon}(A,\sigma) \coprod X_{-\varepsilon}(K)$.

As in the classical case we have a total signature:

Definition 6.20. *Let (A,σ) be an algebra with involution over K. The total signature of any $x \in \widetilde{W}(A,\sigma)$ is the function*

$$\widetilde{\mathrm{sign}}(x) : \widetilde{X}(A,\sigma) \longrightarrow \mathbb{Z}$$

such that $\widetilde{\mathrm{sign}}(x)(I^{\varepsilon}_P) = \widetilde{\mathrm{sign}}^{\varepsilon}_P(x)$ (which does not depend on any choice of labelling for the signature maps).

Proposition 6.21. *Let (A,σ) be an algebra with involution over K. Then for any $x \in \widetilde{W}(A,\sigma)$, the total signature $\widetilde{\mathrm{sign}}(x)$ is a continuous function.*

Proof. By definition, $\widetilde{\mathrm{sign}}(x)^{-1}(\{n\})$ is the intersection of $\widetilde{X}(A,\sigma)$ with the Zariski-closed set $V(x - n\langle 1\rangle)$ in $\mathrm{Spec}(\widetilde{W}(A,\sigma))$, so it is closed in $\widetilde{X}(A,\sigma)$. □

Corollary 6.22. *The topological space $\widetilde{X}(A,\sigma)$ is compact and totally disconnected.*

Proof. Let us consider the function

$$\begin{array}{rccc} F: & \widetilde{X}(A,\sigma) & \longrightarrow & \{-1,0,1\}^{\widetilde{W}(A,\sigma)} \\ & \widetilde{I}^{\varepsilon}_P & \longmapsto & (x \mapsto \tau(\widetilde{\mathrm{sign}}^{\varepsilon}_P(x))) \end{array}$$

where $\tau : \mathbb{Z} \to \{-1,0,1\}$ maps non-zero integers to their sign. Then we claim that F is injective and continuous; since the target space is compact this is a homeomorphism onto its image, which concludes.

The injectivity is clear: if two ideals have the same image, then considering the restriction to elements $\langle a\rangle \in W(K)$ we see that they must correspond to the same $P \in X(K)$, and if the signature maps have the same signs on hermitian forms they are equal.

For continuity, note that F corresponds to the same map

$$\Phi : \widetilde{X}(A,\sigma) \times \widetilde{W}(A,\sigma) \to \{-1,0,1\}$$

as $\tau \circ \widetilde{\mathrm{sign}}$. Since each $\widetilde{\mathrm{sign}}(x)$ is continuous, Φ is continuous for the product topology if we put the discrete topology on $\widetilde{W}(A,\sigma)$, and so F is continuous. □

6.5 Polarizations

One on the main goals in [2] and [3] can be interpreted as the definition of an appropriate total signature that is defined on $X(K)$ instead of $\widetilde{X}(A,\sigma)$ (this is what they call $\mathcal{M}$-signatures and H-signatures).

Definition 6.23. *Let (A,σ) be an algebra with involution over K. If U is an open subset of $X(K)$, a local polarization of (A,σ) over U is a set-theoretical section of $\overline{\pi}$ on U. We write $\mathrm{Pol}_U(A,\sigma)$ for the set of local polarizations over U. If $s \in \mathrm{Pol}_U(A,\sigma)$, we say that $-s \in \mathrm{Pol}_U(A,\sigma)$, such that $-s(P) \neq s(P)$ for all $P \in U$, is the opposite (local) polarization of s.*

When $U = X(K)$ (resp. $X_+(A)$, $X_-(A)$), we speak of a global (resp. orthogonal, symplectic) polarization of (A,σ), and the set of those is denoted by $\mathrm{Pol}(A,\sigma)$ (resp. $\mathrm{Pol}_+(A,\sigma)$, $\mathrm{Pol}_-(A,\sigma)$). A global polarization is also simply called a polarization.

If $s \in \mathrm{Pol}(A,\sigma)$, then for any $x \in \widetilde{W}(A,\sigma)$, the total signature of x relative to s is

$$\widetilde{\mathrm{sign}}^s(x) : X(K) \xrightarrow{s} \widetilde{X}(A,\sigma) \xrightarrow{\widetilde{\mathrm{sign}}(x)} \mathbb{Z}.$$

We also write $\widetilde{\mathrm{sign}}^s_P(x) = \widetilde{\mathrm{sign}}^s(x)(P)$.

Clearly $\mathrm{Pol}(A,\sigma) \simeq \mathrm{Pol}_+(A,\sigma) \times \mathrm{Pol}_-(A,\sigma)$. The way we see things is that a polarization is the choice of a labelling of $\widetilde{\mathrm{sign}}^+_P$ and $\widetilde{\mathrm{sign}}^-_P$, and an orthogonal (resp. symplectic) polarization is such a choice for only the $P \in X_+(A)$ (resp. $X_-(A)$). The way we defined the signature maps shows that a choice of polarization is also equivalent to a choice of Morita equivalence between (A_{K_P}, σ_{K_P}) and (D_P, θ_P) for all $P \in X(K)$, but the global structure of $\widetilde{X}(A,\sigma)$ makes it much more convenient to discuss polarizations. Our goal is to find relevant natural classes of polarizations, or even ideally natural polarizations on various (A,σ).

Remark 6.24. The notion of $\mathcal{M}$-signature in [2] corresponds to an arbitrary (orthogonal/symplectic) polarization.

Remark 6.25. For any polarization s, and any $x \in W(K)$, $\widetilde{\mathrm{sign}}^s(x)$ is the classical total signature $\mathrm{sign}(x) : X(K) \to \mathbb{Z}$.

There are natural symmetries of polarizations that we want to emphasize. Let G be the group of set-theoretical automorphisms of $\overline{\pi}$, and G_c (for *continuous*) the group of topological automorphisms of $\overline{\pi}$. Then G can be identified with the multiplicative group $\mathcal{F}(X(K),\{\pm 1\})$: a function $f : X(K) \to \{\pm 1\}$ acts by swapping the elements of the fiber above $P \in X(K)$ iff $f(P) = -1$. This is also naturally isomorphic to the group $(\mathcal{P}(X(K)), \Delta)$ of subsets of $X(K)$ with the symmetric difference, is we associate f to $f^{-1}(\{-1\})$. Then $G_c \subset G$ corresponds to the continuous functions inside $\mathcal{F}(X(K),\{\pm 1\})$, and to the clopen subsets inside $\mathcal{P}(X(K))$. Clearly G acts, simply transitively, on $\mathrm{Pol}(A,\sigma)$. If $f \in \mathcal{F}(X(K),\{\pm 1\})$, then its action (as an element of G) on the total signatures is $\widetilde{\mathrm{sign}}^{f\cdot s} = f\widetilde{\mathrm{sign}}^{s}$. The function $s \mapsto -s$ corresponds to the constant function -1 in $\mathcal{F}(X(K),\{\pm 1\})$, and to $X(K) \in \mathcal{P}(X(K))$.

Note that since $\overline{\pi}$ is the application of the functor Spec_0 to the inclusion $W(K) \to \widetilde{W}(A,\sigma)$, there is a canonical embedding of the $W(K)$-algebra automorphisms of $\widetilde{W}(A,\sigma)$ in G_c. We call G_a (for *algebraic*) the image of the embedding. The image of the subgroup of standard automorphisms is denoted G_s (for *standard*). This action can also be deduced from the fact that by construction, $(A,\sigma) \mapsto \widetilde{X}(A,\sigma)$ defines a functor from $\mathbf{Br}_h(K)$ to the category of sets above $X(K)$. Note that G_s is naturally a quotient of K^* since standard automorphisms have the form $(\langle a\rangle_\sigma)_*$ for some $a \in K^*$.

We then have
$$G_s \subset G_a \subset G_c \subset G,$$
and if we cannot find a canonical element of $\mathrm{Pol}(A,\sigma)$ for an arbitrary (A,σ) we can at least try to find canonical classes in $\mathrm{Pol}(A,\sigma)/H$ for those various subgroups $H \subset G$ (of course $\mathrm{Pol}(A,\sigma)/G = \{*\}$), or maybe at least in $\mathrm{Pol}_\varepsilon(A,\sigma)/H$ for some ε.

Remark 6.26. Note that by functoriality with respect to $\mathbf{Br}_h(K)$, $\mathrm{Pol}(A,\sigma)/G_s$ only depends on the Brauer class $[A]$.

The topological nature of our spaces makes it very natural to investigate the following class:

Definition 6.27. *We write* $\mathrm{Pol}^c(A,\sigma)$ *for the set of continuous polarizations on* (A,σ)*, that is continuous sections of* $\overline{\pi}$*. Likewise, we have* $\mathrm{Pol}^c_+(A,\sigma)$ *and* $\mathrm{Pol}^c_-(A,\sigma)$*, so that* $\mathrm{Pol}^c(A,\sigma) \simeq \mathrm{Pol}^c_+(A,\sigma) \times \mathrm{Pol}^c_-(A,\sigma)$.

Remark 6.28. By construction, a polarization is the same as a set-theoretic section of $\pi : \mathrm{Spec}(\widetilde{W}(A,\sigma)) \to \mathrm{Spec}(W(K))$ that is compatible with the specialization of points. Then a continuous polarization is the same as a continuous section of π.

Proposition 6.29. *Let* (A,σ) *be an algebra with involution over* K*. A polarization* $s \in \mathrm{Pol}(A,\sigma)$ *is continuous iff for all* $x \in \widetilde{W}(A,\sigma)$*, the total signature* $\widetilde{\mathrm{sign}}^s(x)$ *relative to* s *is continuous on* $X(K)$.

Proof. Since the absolute total signature $\widetilde{\mathrm{sign}}(x)$ is continuous on $\widetilde{X}(A,\sigma)$ (proposition 6.21), clearly if s is a continuous section of $\overline{\pi}$ then the composition $\widetilde{\mathrm{sign}}^s(x)$ is also continuous.

Conversely, assume all $\widetilde{\mathrm{sign}}^s(x)$ are continuous on $\widetilde{X}(A,\sigma)$. Let $D(x) \subset \mathrm{Spec}(\widetilde{W}(A,\sigma))$ be the open subset defined by x (ie the open subscheme defined by the localization at x). By construction, $D_0(x) := D(x) \cap \widetilde{X}(A,\sigma)$ is the subset on which $\widetilde{\mathrm{sign}}(x)$ takes non-zero values, so $s^{-1}(D_0(x))$ is the subset of

$X(K)$ on which $\widetilde{\mathrm{sign}}^s(x)$ takes non-zero values. By hypothesis, it is open in $X(K)$. Since the $D_0(x)$ form an open basis of $\widetilde{X}(A,\sigma)$, this means that s is continuous. $\square$

It follows from the definition of G_c that if $\mathrm{Pol}^c(A,\sigma)$ is not empty, then it is a simply transitive G_c-set, so it defines a class in $\mathrm{Pol}(A,\sigma)/G_c$. Thus we just need to know whether there is one continuous polarization to find all of them. This is strongly related to the study of H-signatures in [3], as we will now investigate.

Definition 6.30. *Let (A,σ) be an algebra with involution over K. If $x \in \widetilde{W}(A,\sigma)$, we write*

$$U(x) = \{P \in X(K) \mid \widetilde{\mathrm{sign}}^+_P(x) \neq \widetilde{\mathrm{sign}}^-_P(x)\}.$$

We call $U(x)$ the principal subset of $X(K)$ defined by x.

We also define $s_x \in \mathrm{Pol}_{U(x)}(A,\sigma)$, called the principal local polarization defined by x, as the unique local polarization such that $\widetilde{\mathrm{sign}}^{s_x}_P(x) > \widetilde{\mathrm{sign}}^{-s_x}_P(x)$ for all $P \in U(x)$.

Proposition 6.31. *Let (A,σ) be an algebra with involution over K Then for any $x \in \widetilde{W}(A,\sigma)$, $U(x)$ is a clopen subset of $X(K)$, and $s_x : U(x) \to \widetilde{X}(A,\sigma)$ is a continuous local polarization of (A,σ) over $U(x)$.*

Proof. Let $\tau : \widetilde{X}(A,\sigma) \to \widetilde{X}(A,\sigma)$ be the function that swaps the elements of every fiber of π. It is continuous, for instance because it is induced by the standard automorphism defined by $\langle -1\rangle_\sigma$. We define $f : \widetilde{X}(A,\sigma) \to \mathbb{Z}^2$ by $f = (\widetilde{\mathrm{sign}}(x), \widetilde{\mathrm{sign}}(x) \circ \tau)$, and

$$S = \{(m,n) \in \mathbb{Z}^2 \mid m \neq n\}, \quad S^+ = \{(m,n) \in \mathbb{Z}^2 \mid m > n\}.$$

Then $U(x) = \pi(f^{-1}(S))$ and $\mathrm{Im}(s_x) = f^{-1}(S^+)$, so $\mathrm{Im}(s_x)$ is closed in $\widetilde{X}(A,\sigma)$ and $U(x)$ is clopen in $X(K)$ (here we use the compacity of $\widetilde{X}(A,\sigma)$). Now if Y is any closed set in $\widetilde{X}(A,\sigma)$, then $s_x^{-1}(Y) = \pi(Y \cap \mathrm{Im}(s_x))$, so it is closed in $X(K)$, which shows that s_x is continuous. $\square$

Then we interpret the results in [3] as:

Theorem 6.32. *Let (A,σ) be an algebra with involution over K. Then for any $x_1,\dots,x_n \in W_\varepsilon(A)$, there exists $x \in W_\varepsilon(A,\sigma)$ such that $U(x_1)\cup\cdots\cup U(x_n) = U(x)$. In particular, there exists $x \in W_\varepsilon(A,\sigma)$ such that $U(x) = X_\varepsilon(A)$, so there exist global continuous polarizations, and $\widetilde{X}(A,\sigma) \approx X(K)\coprod X(K)$ as topological spaces, with $\overline{\pi}$ being the canonical projection.*

Furthermore, the class of global principal polarizations is a transitive G_c-set, thus it is exactly $\mathrm{Pol}^c(A,\sigma)$.

Proof. The existence of $x \in W_\varepsilon(A,\sigma)$ such that $\bigcup U(x_i) = U(x)$ is a reformulation of [3, 3.1]. The existence of $x \in W_\varepsilon(A,\sigma)$ such that $U(x) = X_\varepsilon(A)$ follows by compacity, since lemma 6.13 shows that the $U(x_i)$ form an open cover of $X_\varepsilon(A)$. Since $\mathrm{Pol}^c_+(A,\sigma)$ and $\mathrm{Pol}^c_-(A,\sigma)$ are non-empty, so is $\mathrm{Pol}^c(A,\sigma)$. If $s \in \mathrm{Pol}(A,\sigma)$, then $\widetilde{X}(A,\sigma) = \mathrm{Im}(s)\coprod \mathrm{Im}(-s)$, and if s is continuous, $\mathrm{Im}(s)$ and $\mathrm{Im}(-s)$ are homeomorphic to $X(K)$.

The fact that the principal polarizations are a transitive G_c-set is a reformulation of [3, 3.3]. Since they are included in $\mathrm{Pol}^c(A,\sigma)$ and $\mathrm{Pol}^c(A,\sigma)$ is also a transitive G_c-set, we can conclude. $\square$

Remark 6.33. With this framework, the weaker lemma 6.13 simply states that the $U(x)$ with $x \in W_\varepsilon(A,\sigma)$ form a open cover of $X_\varepsilon(A)$, whichs shows that $\widetilde{X}(A,\sigma)$ is a double cover of $X(K)$, since it has local trivialization. The notion of H-signatures in [2] corresponds to taking a finite open cover of $X_\varepsilon(A)$ by principal subsets (which exists by compacity). In general, H-signatures correspond to continuous polarizations.

Given that $\mathrm{Spec}(\widetilde{W}(A,\sigma))$ is not only a topological space but a scheme, we also have another natural class of polarizations: if $\rho : \widetilde{W}(A,\sigma) \to W(K)$ is a retraction (see definition 6.6), then applying the Spec functor gives a scheme morphism $\rho^* : \mathrm{Spec}(W(K)) \to \mathrm{Spec}(\widetilde{W}(A,\sigma))$, so in particular a continuous polarization. We call polarizations of this form *algebraic polarizations* of (A,σ), and we write $\mathrm{Pol}^a(A,\sigma)$. Similarly, orthogonal (resp. symplectic) retractions define the set $\mathrm{Pol}^a_+(A,\sigma)$ (resp. $\mathrm{Pol}^a_-(A,\sigma)$) of algebraic orthogonal (resp. symplectic) polarizations of (A,σ). There is an obvious injection $\mathrm{Pol}^a(A,\sigma) \subset \mathrm{Pol}^a_+(A,\sigma) \times \mathrm{Pol}^a_-(A,\sigma)$, but it is not clear if it is surjective in general. The existence of an algebraic (global, orthogonal or symplectic) polarization of (A,σ) obviously depends only on the Brauer class $[A]$.

Remark 6.34. By construction, G_a acts on $\mathrm{Pol}^a_+(A,\sigma)$ and $\mathrm{Pol}^a_-(A,\sigma)$, and it acts transitively on $\mathrm{Pol}^a(A,\sigma)$. In particular, there is a well-defined "algebraic" element in $\mathrm{Pol}(A,\sigma)/G_a$, and the sets $\mathrm{Pol}^a(A,\sigma)/G_s$, $\mathrm{Pol}^a_+(A,\sigma)/G_s$ and $\mathrm{Pol}^a_-(A,\sigma)/G_s$ are well-defined, and they only depend on the Brauer class $[A]$.

Example 6.35. If A is split, there are algebraic polarizations of (A,σ), by example 6.7.

Example 6.36. According to proposition 6.8 and remark 6.9, there is always a canonical algebraic symplectic polarization of $(\mathbf{H}_K,\gamma)$, and if K is Pythagorean there is a canonical global algebraic polarization. On the other hand, $(\mathbf{H}_K,\gamma)$ does not have algebraic polarizations if the Pythagoras number of K is at least 3.

We do not know of any other cases where algebraic polarizations exist, and it would be interesting to characterize the Brauer classes for which it is the case.

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